Lorentzian Geometry & Hyperbolic PDEs

Contents

Part I: Lorentzian Geometry

Lorentzian metric:

$(\mathbb{R}^d, g)$ g symmetric non-degenerate quadratic form (2-form)

i) $(\forall X, Y \in \mathbb{R}^d)$ $g(X,Y) = g(Y,X)$

ii) $(\forall X \in \mathbb{R}^d)$ $g(X, \cdot) = 0 \implies X = 0$

$X = (x_1, \dots, x_d)^T$, $Y = (y_1, \dots, y_d)^T$, $\underline{g(X,Y) = X^T \cdot \hat{g} \cdot Y}$, $\hat{g} \in GL(n)$

i) $\hat{g} = \hat{g}^T$

ii) $\det \hat{g} \neq 0$

Spectral Theorem $\Rightarrow (\exists R \in O(n))\ R^T \hat{g} R = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$, $\{\lambda_i\}_{i=1}^{d} \subseteq \mathbb{R}\setminus\{0\}$. WLoG $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0 > \lambda_{s+1} \geq \dots \geq \lambda_d$

Signature of g: $(\underbrace{++\cdots+}_{s}\ \underbrace{--\cdots-}_{d-s})$

Riemannian: $(++\cdots+)$ $\quad s=d$

Lorentzian: $(+-\cdots-)$ $\quad s=1$

Warning: sometimes $(-++\cdots+)$

Convention: $\mathbb{R}^{1,d} = (\mathbb{R}^{1+d}, \eta)$, $\hat{\eta} = \mathrm{diag}(1, \underbrace{-1, \dots, -1}_{d})$

(pseudo-) OrthoNormal Basis (ONB)

$\{X_i\}_{i=0}^{d} \subseteq \mathbb{R}^{1,d}$ s.t. $\underline{\eta(X_0,X_0)=1} \wedge \eta(\underline{X_1,X_1) = \cdots = \eta(X_d,X_d)}=-1$.

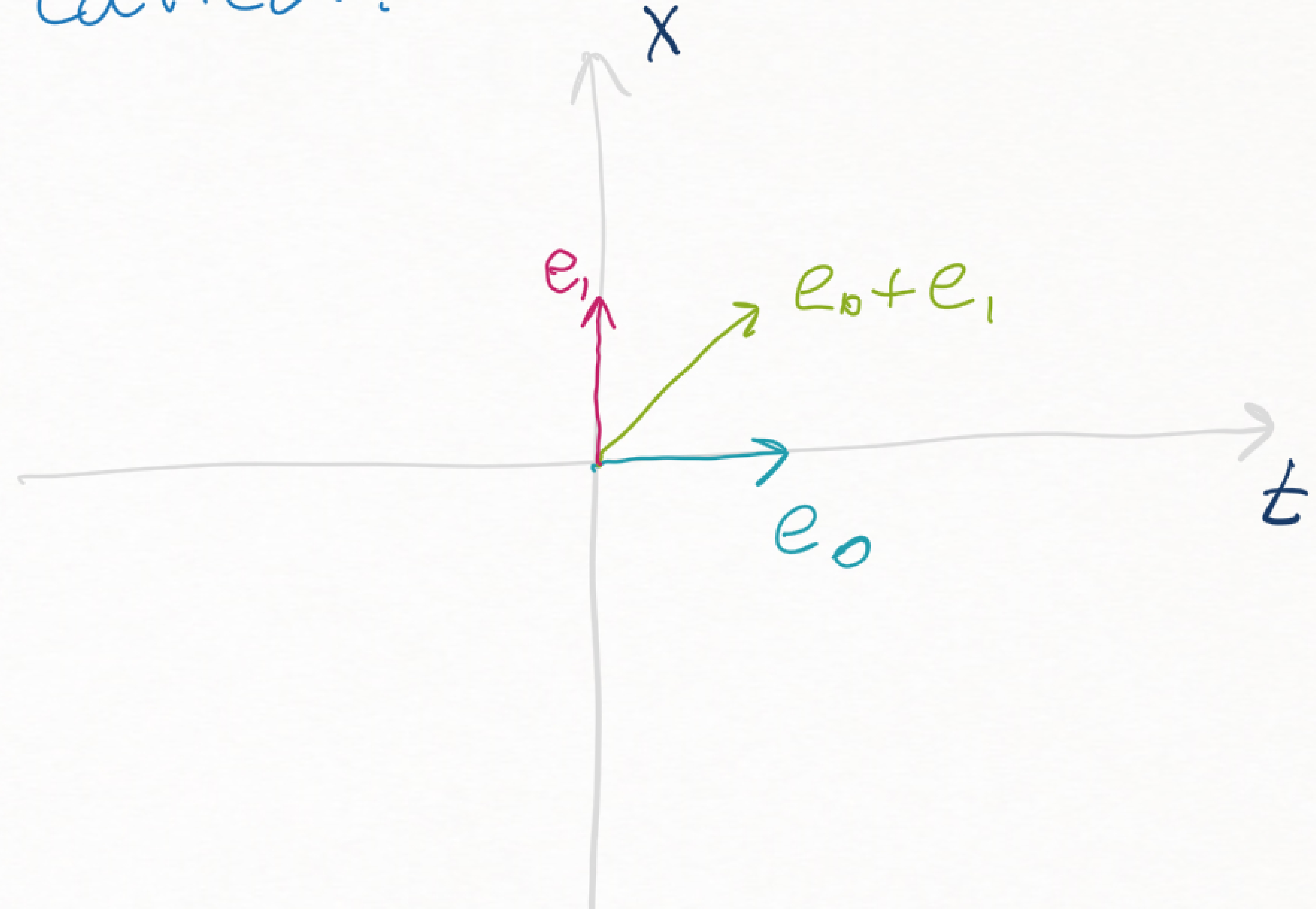
Example: $\{e_i\}_{i=0}^{d}$, $e_0=(1,0,\dots,0)$, $e_1=(0,1,\dots,0)$, $\dots$, $e_d=(0,\dots,1)$

$\eta(e_0,e_0) = \underline{e_0^T \hat{\eta} e_0 = 1}$, $\eta(e_i,e_i) = \underline{e_i^T \hat{\eta} e_i = -1}$, $i=1,\dots,d$.

A vector $X \in \mathbb{R}^{1,d}$ is called:

- Spacelike: $\eta(X,X)<0$
- Light like/null: $\eta(X,X)=0$
- Timelike: $\eta(X,X)>0$

$$\left.\begin{array}{l}\eta(X,X)=0\\ \eta(X,X)>0\end{array}\right\}\text{Causal}$$



Lorentzian Manifold (Spacetime)

(M,g) — M – connected C^∞ manifold

$g \in C^\infty(T_* M^{\otimes 2})$ symmetric type $(0,2)$ tensor s.t.

$(\forall p \in M)$ $g(p)$ is a Lorentzian form on T_pM.

g – Lorentzian metric

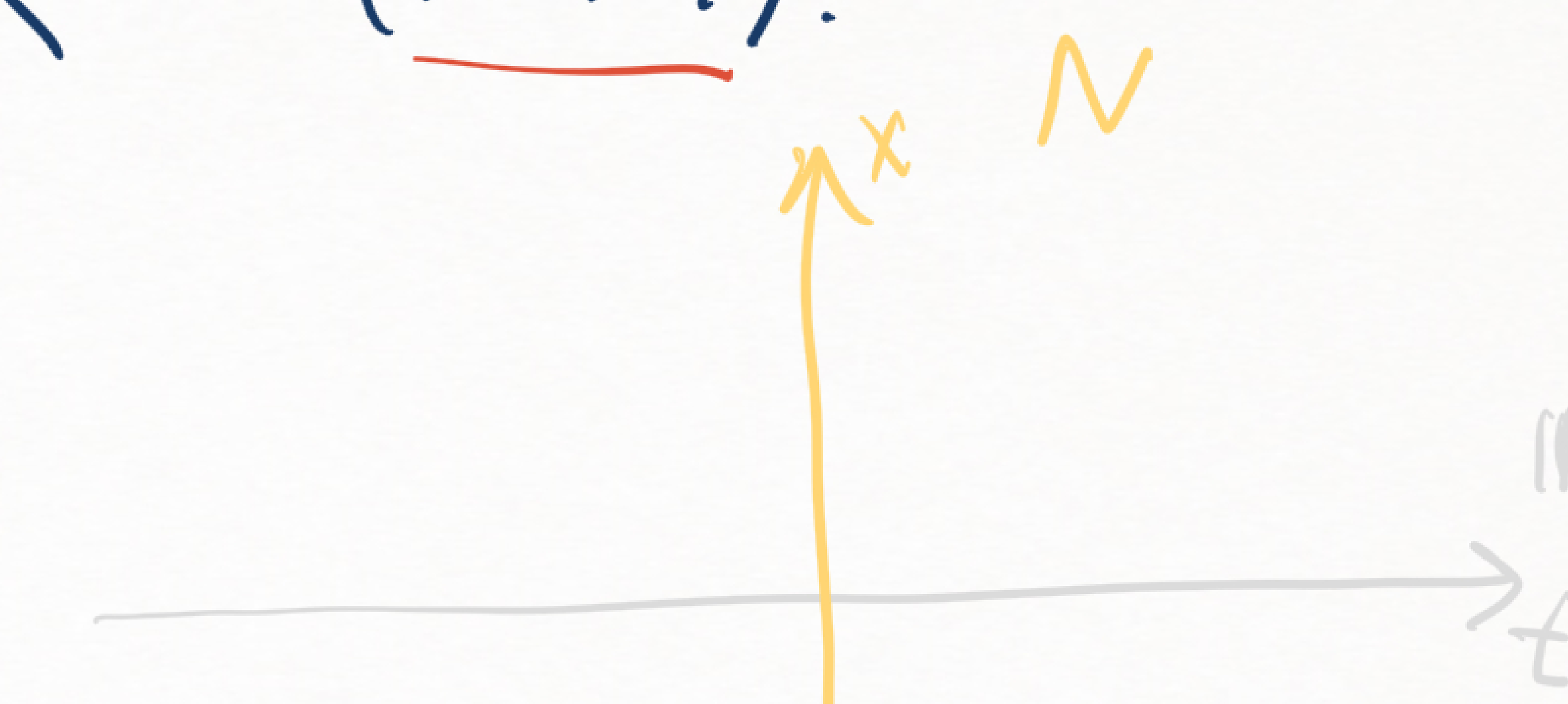
Example: Minkowski spacetime $\mathbb{R}^{1,d} = (\underline{\mathbb{R}^{1+d}, \eta})$.

Example: $(\underline{N, h})$ – Riemannian manifold

$M \triangleq \underline{\mathbb{R} \times N}$

$g \triangleq \underline{1 \oplus -h}$

$(\underline{M, g})$ – Lorentzian manifold



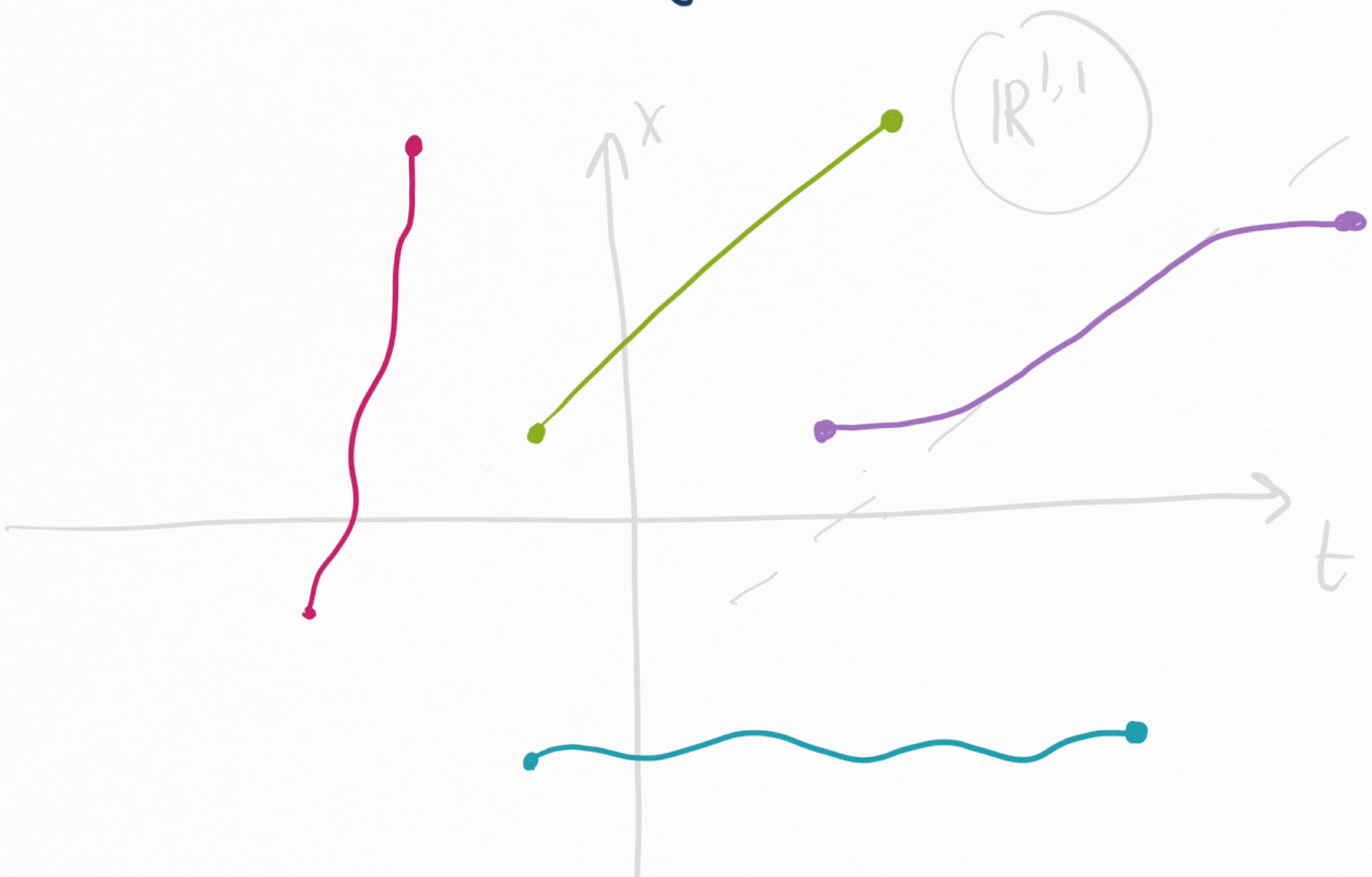
$$g = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\hat{h} \end{array}\right)$$

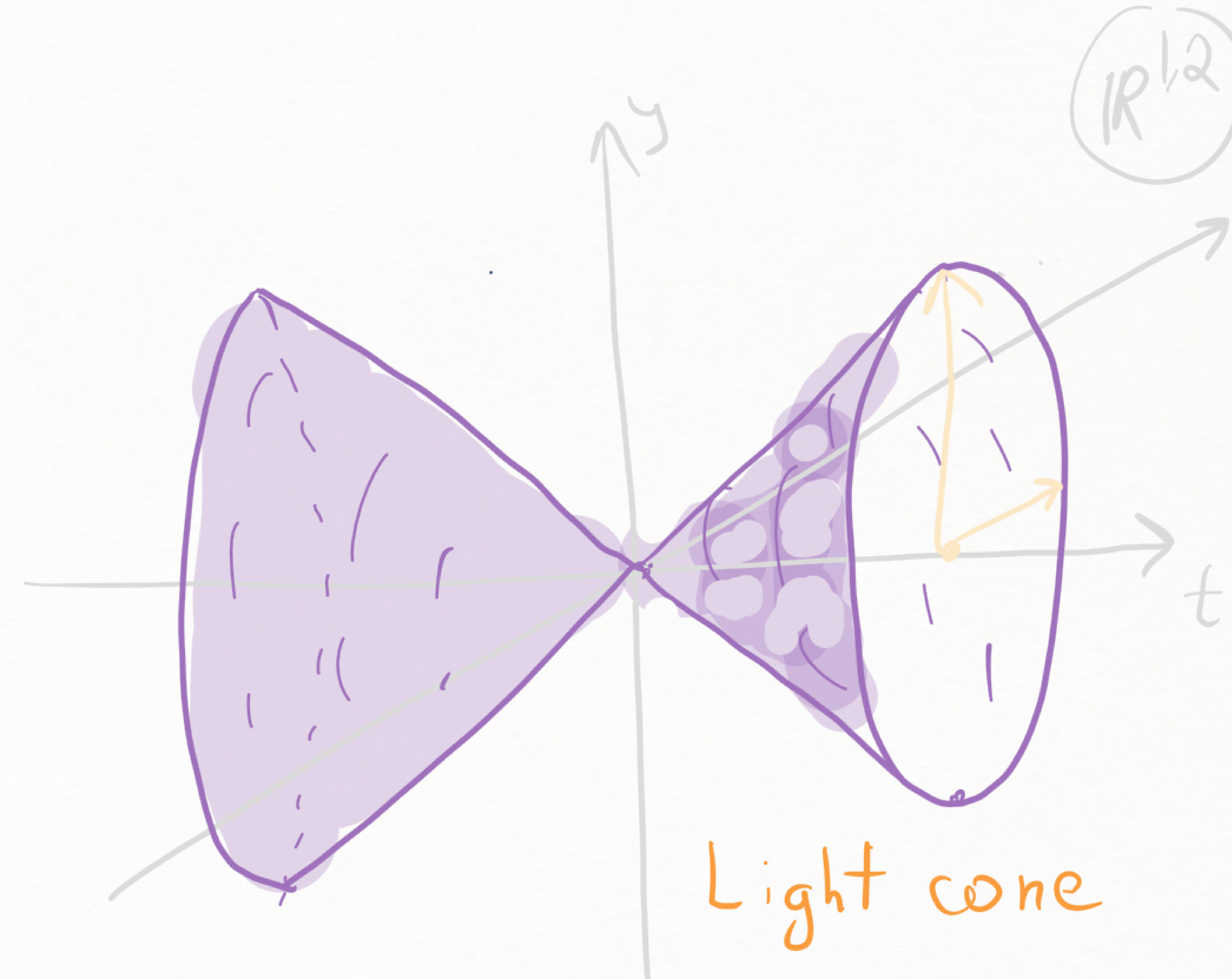
Causal Structure:

A curve $\gamma \in C^1([0,1], M)$ is called

- Spacelike $\quad g(\dot\gamma(s), \dot\gamma(s)) < 0, \ \forall s \in [0,1]$
- Lightlike/null $\quad g(\dot\gamma(s), \dot\gamma(s)) = 0, \ \forall s \in [0,1]$
- Timelike $\quad g(\dot\gamma(s), \dot\gamma(s)) > 0, \ \forall s \in [0,1]$

Causal $\quad g(\dot\gamma(s), \dot\gamma(s)) \geq 0, \ \forall s \in [0,1]$





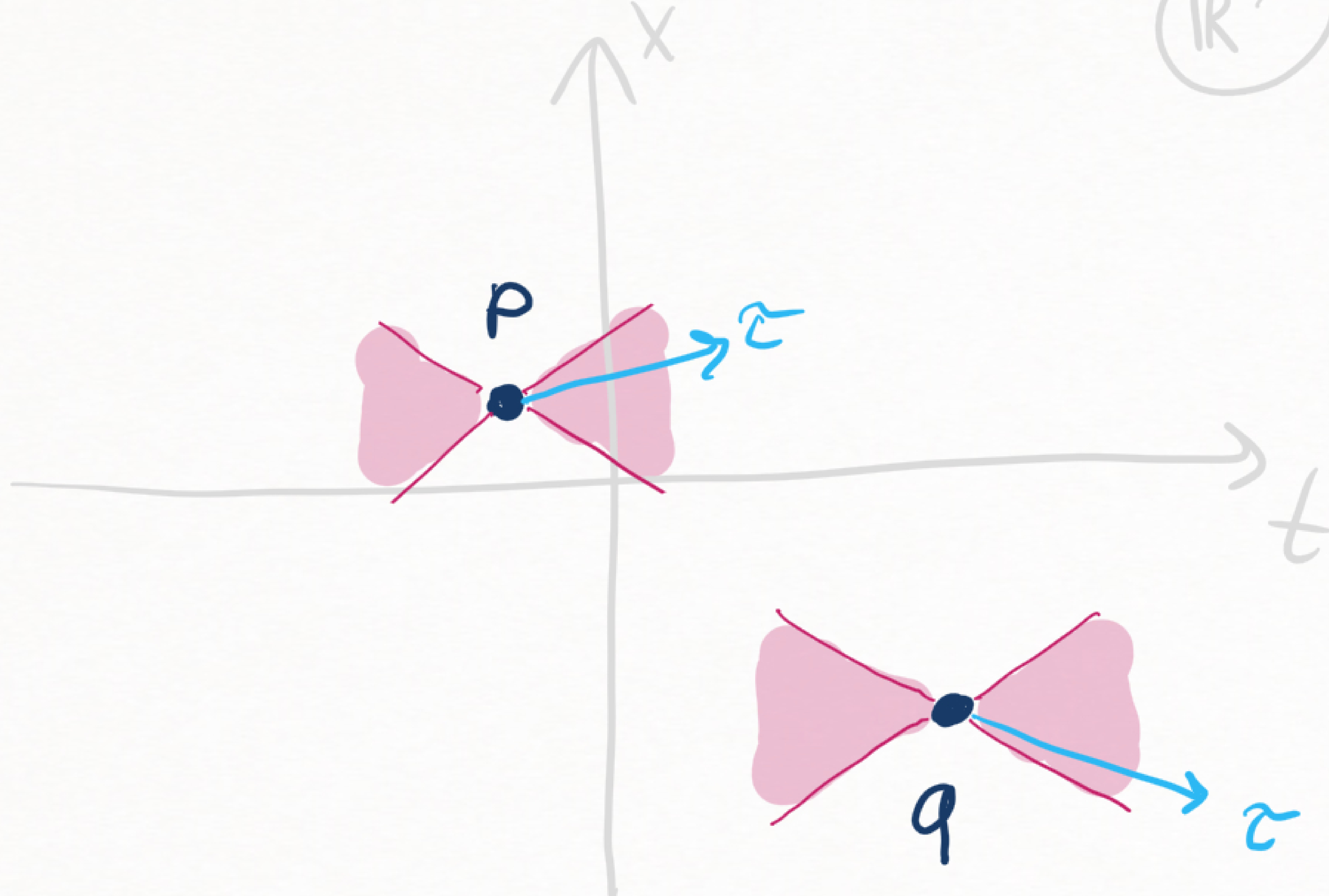
Time Orientation (future/past)

A vector field $X \in C^\infty(TM)$ is called

Spacelike	$g(X(p), X(p)) < 0 \quad, \forall p \in M$
Light like/null	$g(X(p), X(p)) = 0 \quad, \forall p \in M$
Timelike	$g(X(p), X(p)) > 0 \quad, \forall p \in M$

Time Orientation: timelike vector field $\tau \in C^\infty(TM)$, $g(\tau, \tau) > 0$.

$\mathbb{R}^{1,1}$



A causal vector $X \in T_pM$ is called

Future-directed	$g(X, \tau(p)) > 0$
Past-directed	$g(X, \tau(p)) < 0$

By definition, τ is future-directed.

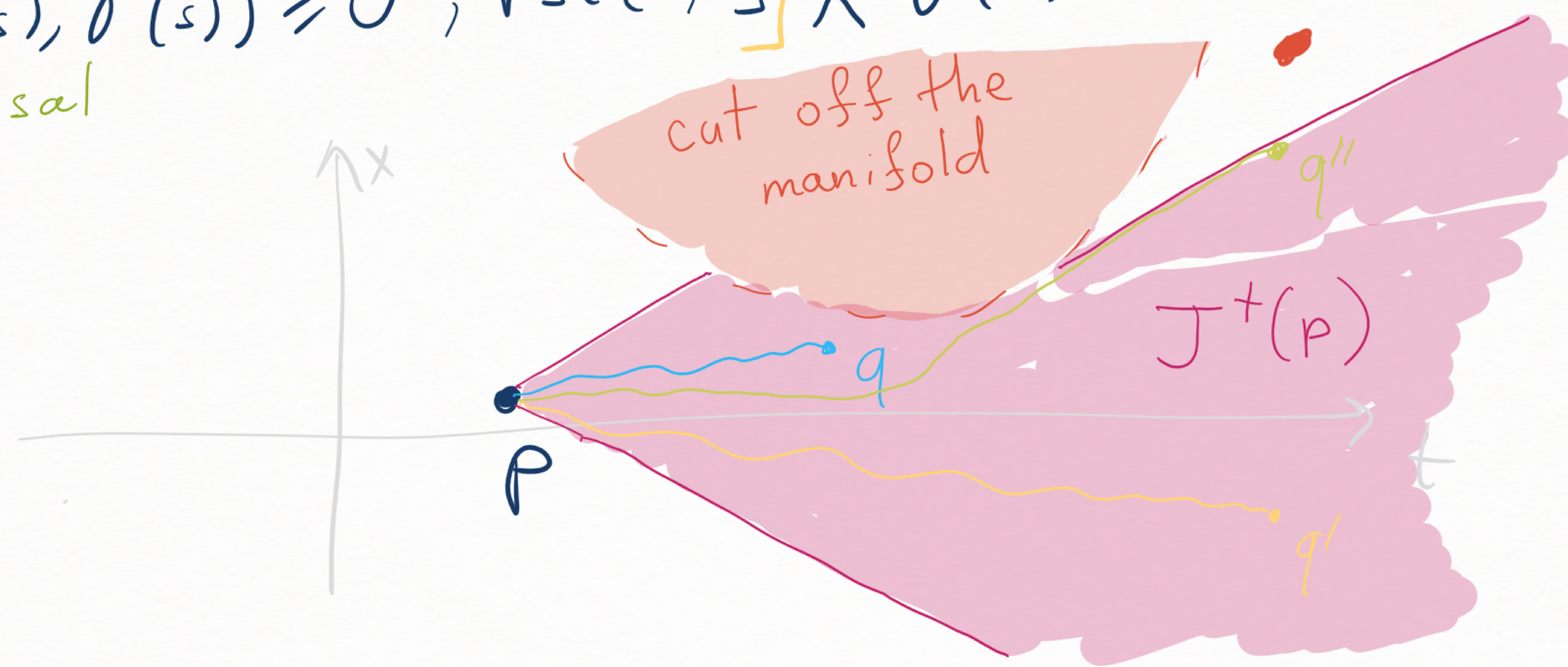
Causal/Chronological future/past:

A causal curve $\gamma \in C^1([0,1], M)$ is called

Future/past-directed $\dot\gamma(s)$ is future/past-directed, $\forall s \in [0,1]$.

Causal Future of $p \in M$

$$J^+(p) = \{ q \in M \mid \exists \gamma \in C^1([0,1], M) \text{ s.t. } [\underbrace{g(\dot\gamma(s), \tau(\gamma(s))) > 0}_{\text{future-directed}} \wedge \underbrace{g(\dot\gamma(s), \dot\gamma(s)) \geq 0}_{\text{causal}}, \forall s \in [0,1]] \wedge \gamma(0) = p \wedge \gamma(1) = q \} \cup \{p\}.$$



Causal Past of $p \in M$

$$J^-(p) = \Big\{ q \in M \;\Big|\; \exists\, \gamma \in C^1([0,1], M) \text{ s.t. } \Big[\underbrace{g(\dot\gamma(s), \tau(\gamma(s))) < 0}_{\text{past-directed}} \wedge \underbrace{g(\dot\gamma(s), \dot\gamma(s)) \geq 0}_{\text{causal}},\ \forall s \in [0,1] \Big] \wedge \gamma(0) = p \wedge \gamma(1) = q \Big\} \cup \{p\}$$

Chronological Future of $p \in M$

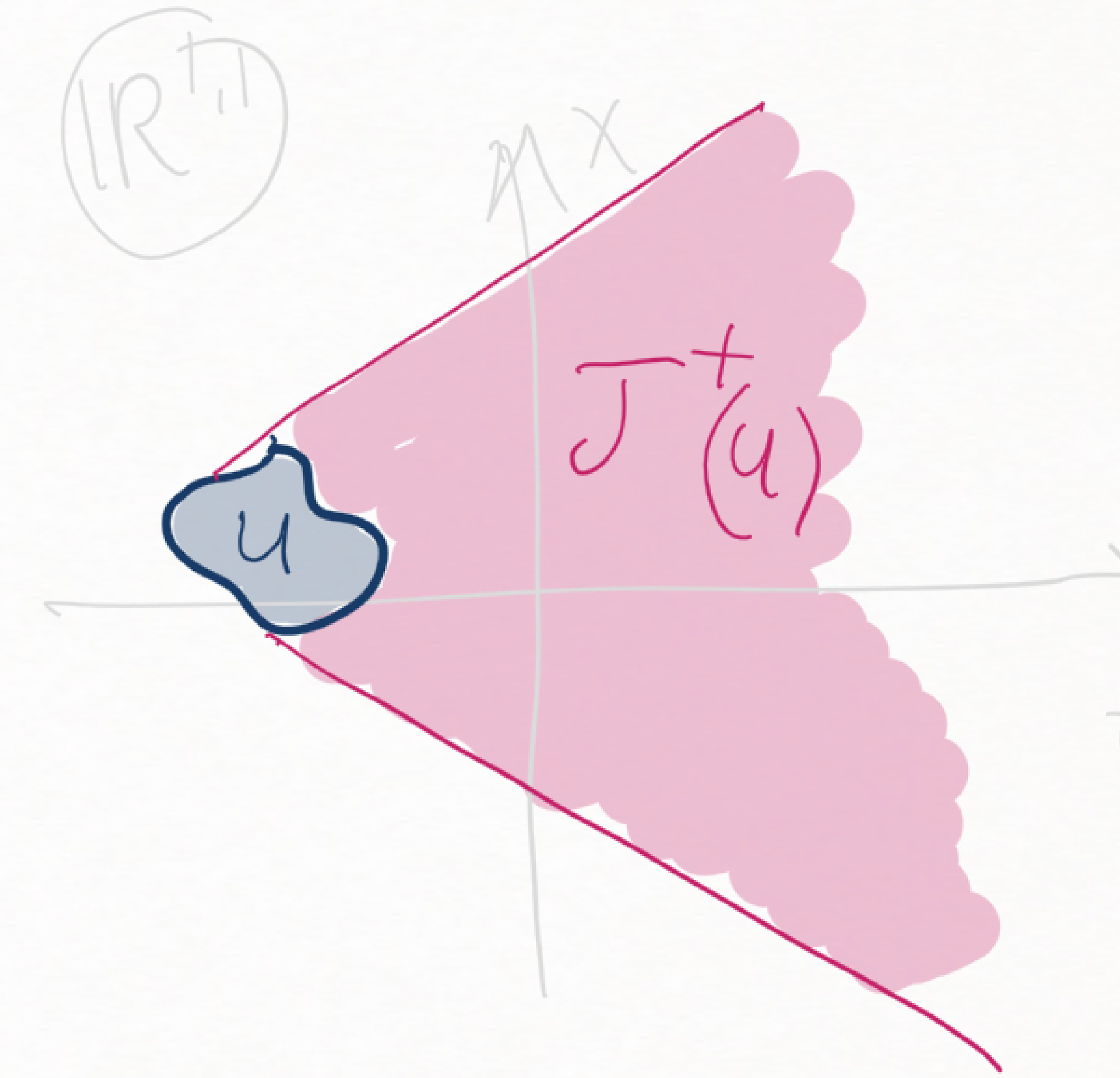
$$I^+(p) = \Big\{ q \in M \;\Big|\; \exists\, \gamma \in C^1([0,1], M) \text{ s.t. } \Big[\underbrace{g(\dot\gamma(s), \tau(\gamma(s))) > 0}_{\text{future-directed}} \wedge \underbrace{g(\dot\gamma(s), \dot\gamma(s)) > 0}_{\text{timelike}},\ \forall s \in [0,1] \Big] \wedge \gamma(0) = p \wedge \gamma(1) = q \Big\}$$

Chronological Past of $p \in M$

$$I^-(p) = \Big\{ q \in M \;\Big|\; \exists\, \gamma \in C^1([0,1], M) \text{ s.t. } \Big[\underbrace{g(\dot\gamma(s), \tau(\gamma(s))) < 0}_{\text{past-directed}} \wedge \underbrace{g(\dot\gamma(s), \dot\gamma(s)) > 0}_{\text{timelike}},\ \forall s \in [0,1] \Big] \wedge \gamma(0) = p \wedge \gamma(1) = q \Big\}$$

Causal/Chronological future/past of $U \subseteq M$

$$J^\pm(U) = \bigcup_{q\in U} J^\pm(q), \quad \mathcal{I}^\pm(U) = \bigcup_{q\in U} \mathcal{I}^\pm(U)$$

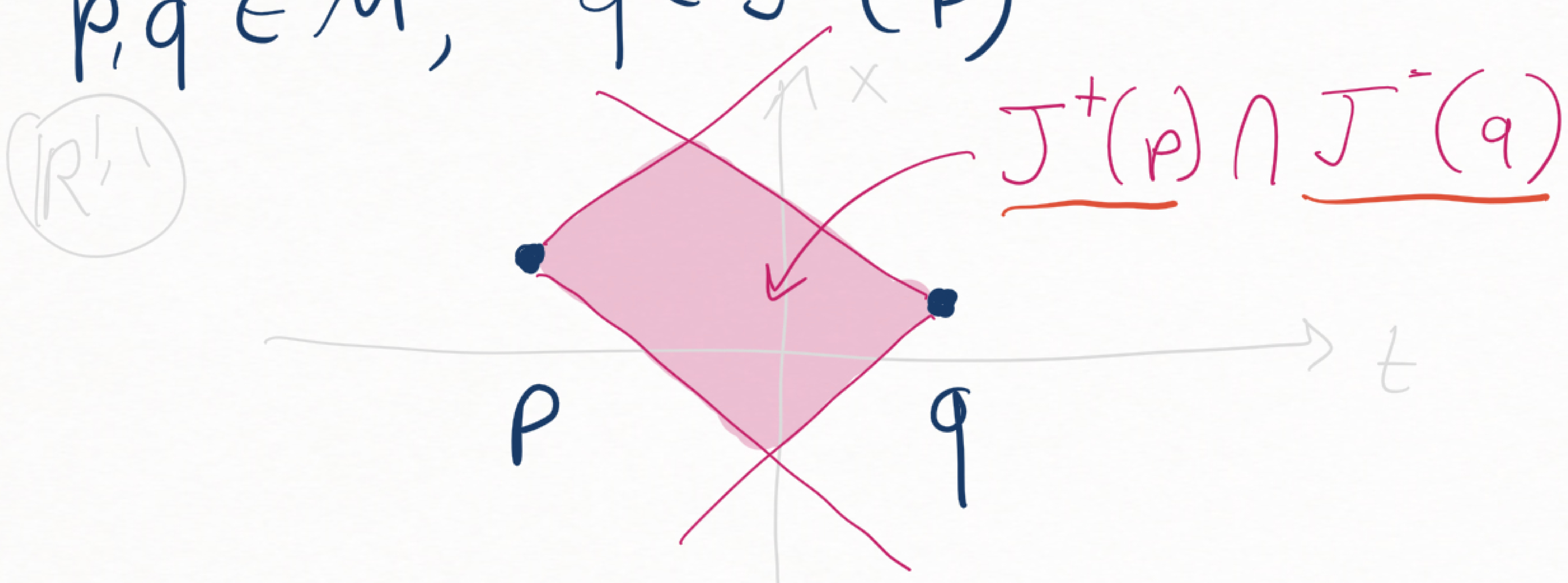


Properties:

- $\mathcal{I}^\pm(U) = \overset{\circ}{J^\pm}(U)$ (interior)
- $J^\pm(U)$ need not be closed

Causal diamond

$p, q \in M$, $q \in J^+(p)$



Curvature and Geodesics

Definitions remain valid independent of signature:

Levi-Civita connection (covariant derivative)

$$\nabla g = 0 \quad \wedge \quad \underbrace{T(\nabla) = 0}_{\text{torsion-free}} \qquad \nabla_X = X + \underbrace{\Gamma_X}_{\text{Christoffel symbols}} \qquad \forall X \in C^\infty(TM)$$

Riemannian curvature

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \forall X, Y \in C^\infty(TM)$$

Geodesics

$$\nabla_{\dot\gamma} \dot\gamma = 0$$

Null geodesics — light

Timelike geodesics — "freely falling" massive objects

Local coordinates

$(t, x_1, \dots, x_d)$ $\quad \partial_t$ timelike, $\{\partial_{x_i}\}_{i=1}^{d}$ spacelike

$$g = g_{00}\, dt \otimes dt + \sum_{i=1}^{d} g_{0i} \underline{(dt \otimes dx_i + dx_i \otimes dt)} + \sum_{i,j=1}^{d} g_{ij}\, dx_i \otimes dx_j$$

Quadratic form notations

$$ds^2 = g_{00}\, dt^2 + 2 \sum_{i=1}^{d} g_{0i}\, dt\, dx_i + \sum_{i,j=1}^{d} g_{ij}\, dx_i\, dx_j$$

$$ds^2 = (dt\ dx_1 \cdots dx_d) \cdot \left(\hat{g} \right) \cdot \begin{pmatrix} dt \\ dx_1 \\ \vdots \\ dx_d \end{pmatrix}$$

Examples of spacetimes

Minkowski $M = \mathbb{R}^{1+d}$ $\quad g(t,x) = 1 \oplus -\mathbb{1}_d$ $\quad ds^2 = dt^2 - |d\vec{x}|^2$

Ultrastatic $M = \mathbb{R} \times \Sigma$ $\quad g(t,x) = 1 \oplus -h(x)$ $\quad ds^2 = dt^2 - \sum_{i,j=1}^{d} h(x)^{ij} dx_i dx_j$

(Σ, h) - Riemannian manifold

Warped product

$M = N \times Q$ $\quad g(t,x,y) = \tilde{g}(t,x) \oplus -a(t,x)^2 \cdot h(y)$

$(N, \tilde{g})$ - Lorentzian manifold

(Q, h) - Riemannian manifold

$a \in C^\infty(N, (0,+\infty))$ - warp function

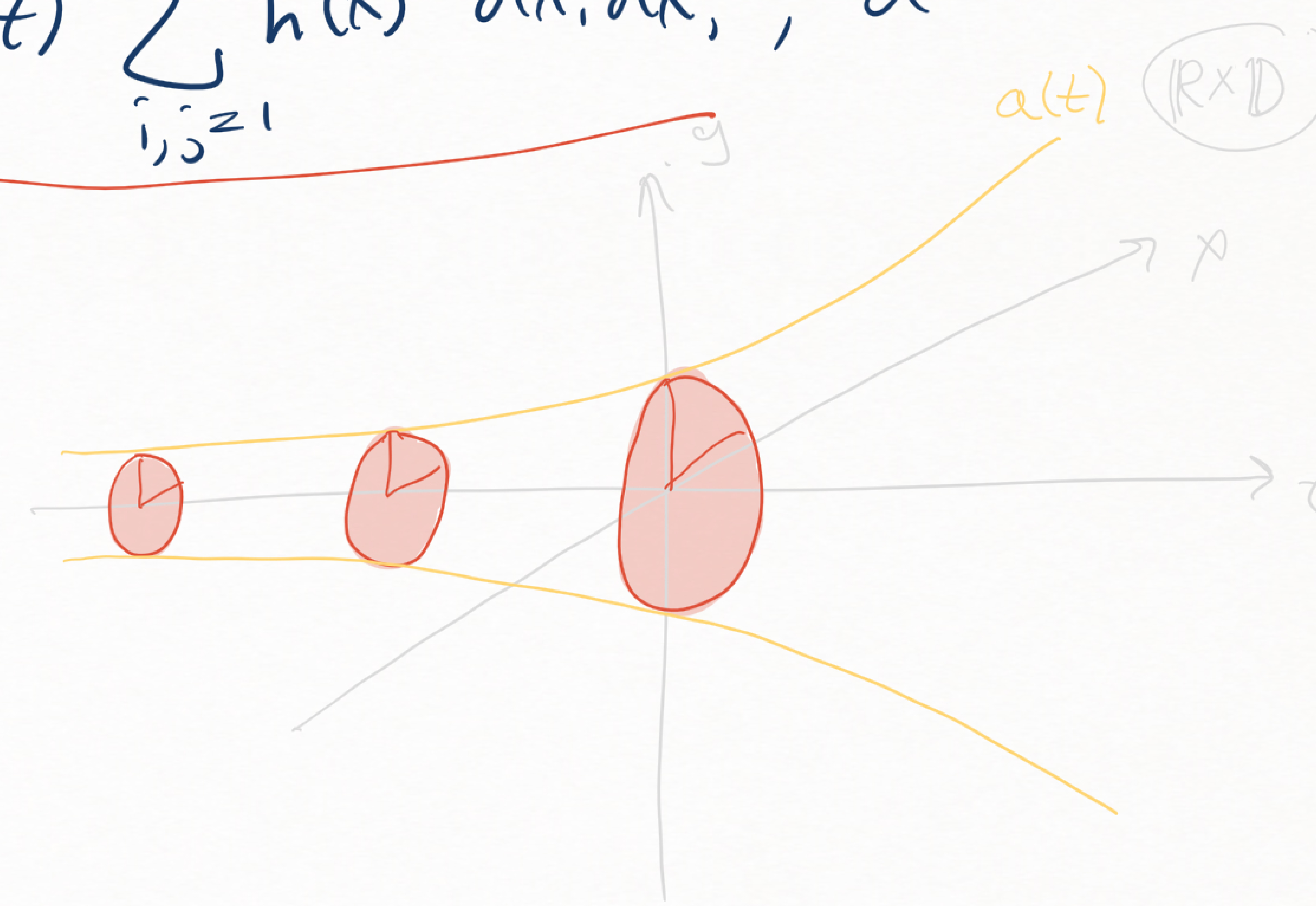
Examples of spacetimes (cont.)

An extreme case $(N, \tilde{g})$ – 1-dim. spacetime ($d=0$, no space)

(Q, h) – Riemannian manifold

$$M = N \times Q \qquad g(t,x) = \beta(t)^2 \oplus - a(t)^2 h(x) \qquad \beta, a \in C^\infty(N, (0,+\infty))$$

$$ds^2 = \beta(t)^2 dt^2 - a(t)^2 \sum_{i,j=1}^{d} h(x)^{ij} dx_i dx_j \,, \quad d = \dim Q.$$

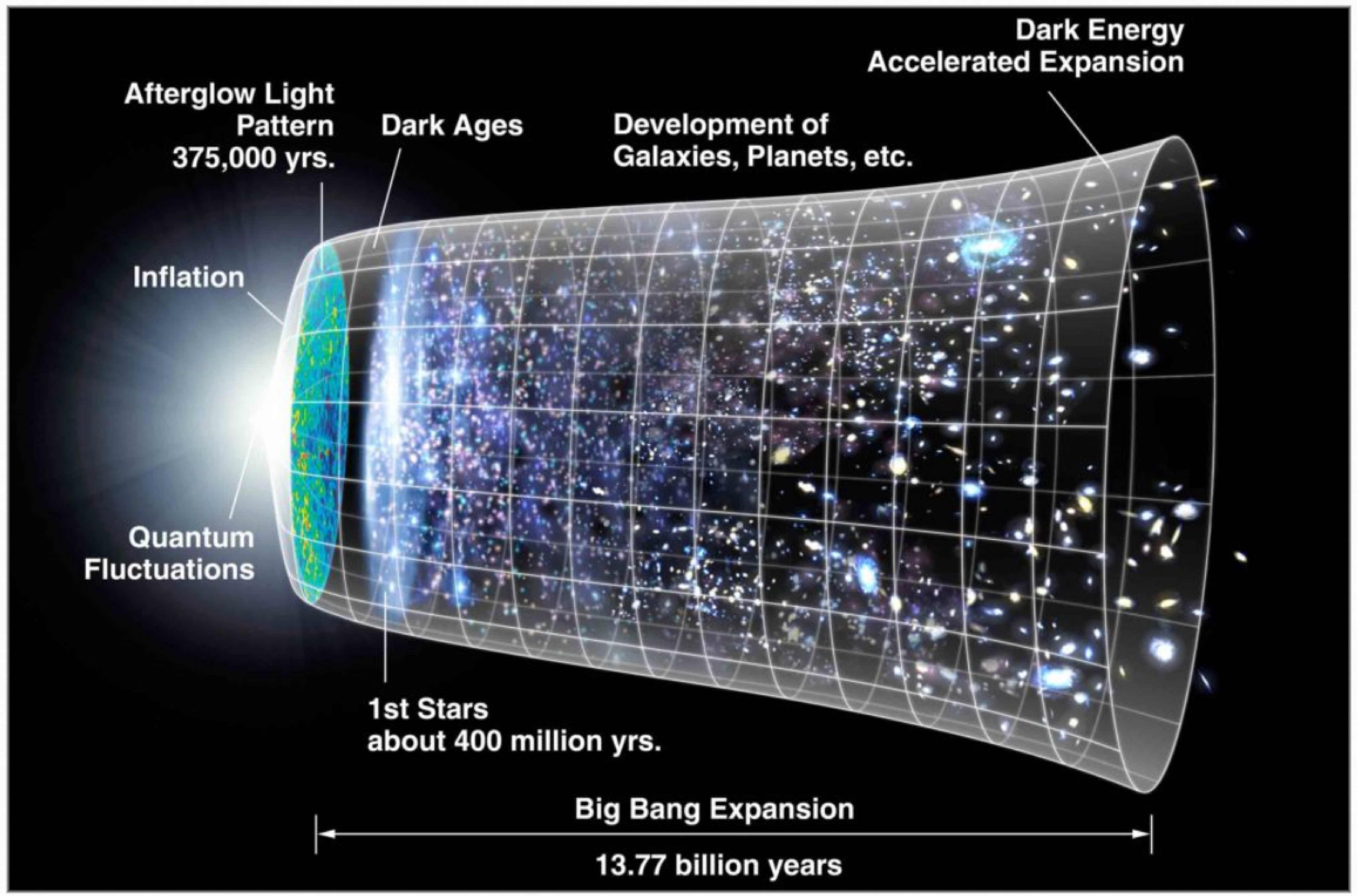


Time reparameterization

$$T(t) = \int_0^t \beta(\tau)\, d\tau \,, \quad A(T) = a(t)$$

$$g(T,x) = 1 \oplus - A(T)^2 h(x)$$

A typical Big Bang spacetime



Google search "cosmology". Subject to copyright, I suppose.

Isometries

Defined in the same way as for Riemannian manifolds.

$$\mathrm{Iso}(M,g) = \{\psi \in \mathrm{Diffeo}(M) \mid d\psi_* g = g\} \quad \text{Lie group}$$

$G' \subseteq \mathrm{Iso}(M,g)$ a Lie subgroup, G' acts on (M,g) by isometries.

Left action notation: $(\forall a \in G')(\forall x \in M)\ \psi_a(x) = a \cdot x$

$$0 \leqslant \dim \mathrm{Iso}(M,g) \leqslant \frac{(d+1)(d+2)}{2}$$

Minkowski spacetime

$$(M,g) = (\mathbb{R}^{1+d}, \eta) = \mathbb{R}^{1,d}$$

$$\mathrm{Iso}(\mathbb{R}^{1,d}) = \underline{\mathbb{R}^{d+1}} \rtimes \underline{O(1,d)}$$ Poincaré group

$O(1,d)$ Lorentz group

$G = \mathrm{Iso}(\mathbb{R}^{1,d})$ acts transitively on $\mathbb{R}^{1,d}$, $G.p = M$, $\forall p \in M$

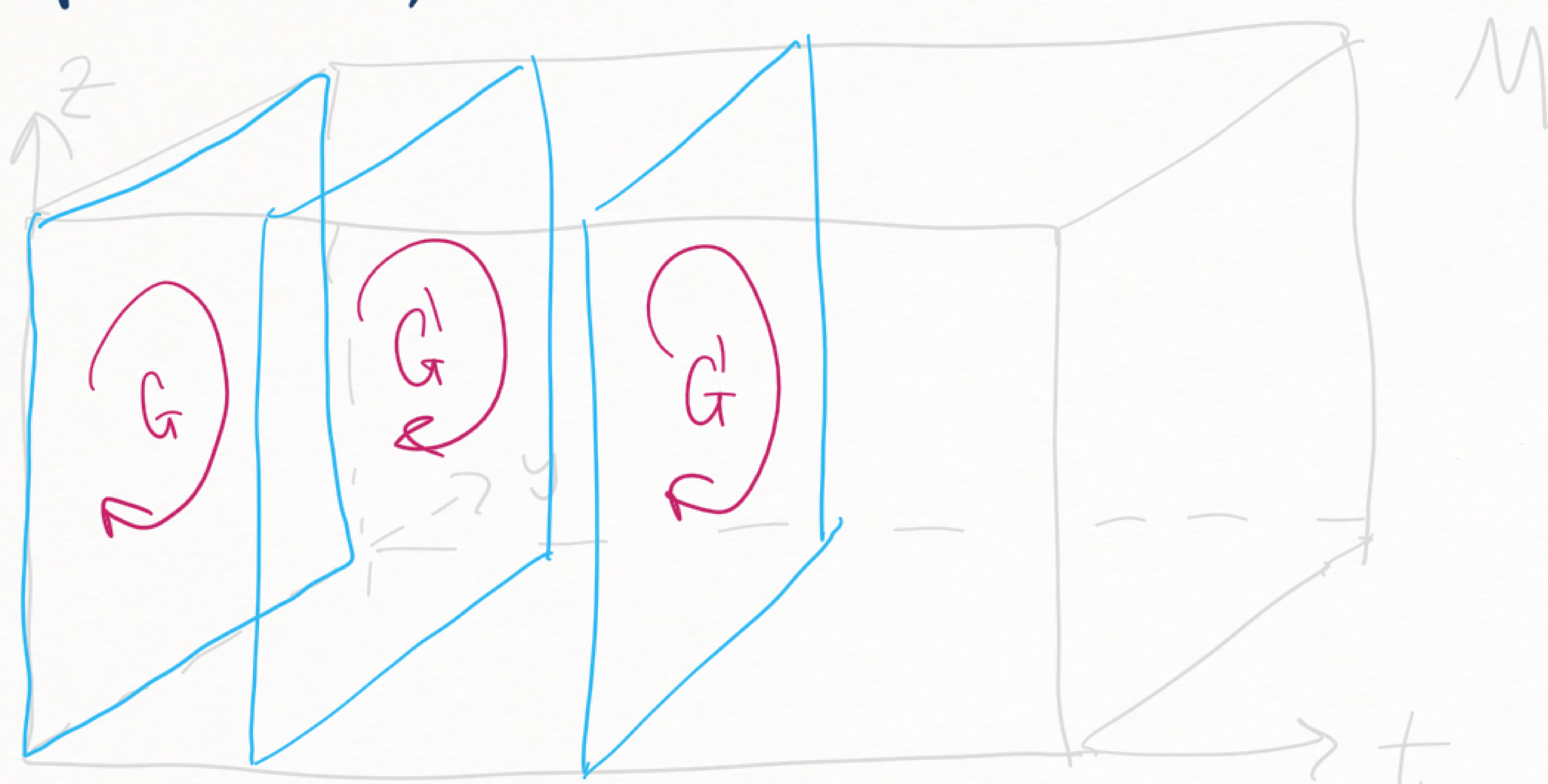
Wigner's classification of elementary particles of nature based on the rep. theory of $\mathrm{Iso}(\mathbb{R}^{1,d})$.

Homogeneous Cosmological Spacetimes

(M,g) spacetime, G Lie group

Spatial homogeneity Let G act on (M,g) by isometries so that

$(\forall p \in M)$ $G\cdot p \subseteq M$ spacelike and $\dim G\cdot p = d$ $(= \dim M - 1)$



$\Sigma = G\cdot p$, $h = g|_\Sigma$

(Σ, h) G-homogeneous Riemannian manifold

Examples

FRW spacetimes $\quad M = \mathbb{R} \times \Sigma, \quad \Sigma \in \{\mathbb{R}^3, S^3\}$

$$g(t,x) = 1 \oplus -a(t)^2 h(x)$$

$(\Sigma, h) = \quad E(3)_0 / SO(3) \quad$ Flat, Euclidean universe $\quad K=0$

$\qquad SO(4)/SO(3) \quad$ Closed universe $\quad K>0$

$\qquad SO(1,3)_0/SO(3) \quad$ Open universe $\quad K<0$

More generally: $g(t,x) = \beta(t)^2 \oplus -h_t(x)$, $\quad (\Sigma, h_t)$ – G-hom. Riem. man. $\forall t \in \mathbb{R}$.

$$\boxed{M = \mathbb{R} \times \Sigma}$$

Inextendible Curves

A curve $\tilde{\gamma} \in C^1((\tilde{\alpha}, \tilde{\beta}), M)$ is called an extension of a curve $\gamma \in C^1((\alpha, \beta), M)$ if $\gamma((\alpha, \beta)) \subsetneq \tilde{\gamma}((\tilde{\alpha}, \tilde{\beta}))$

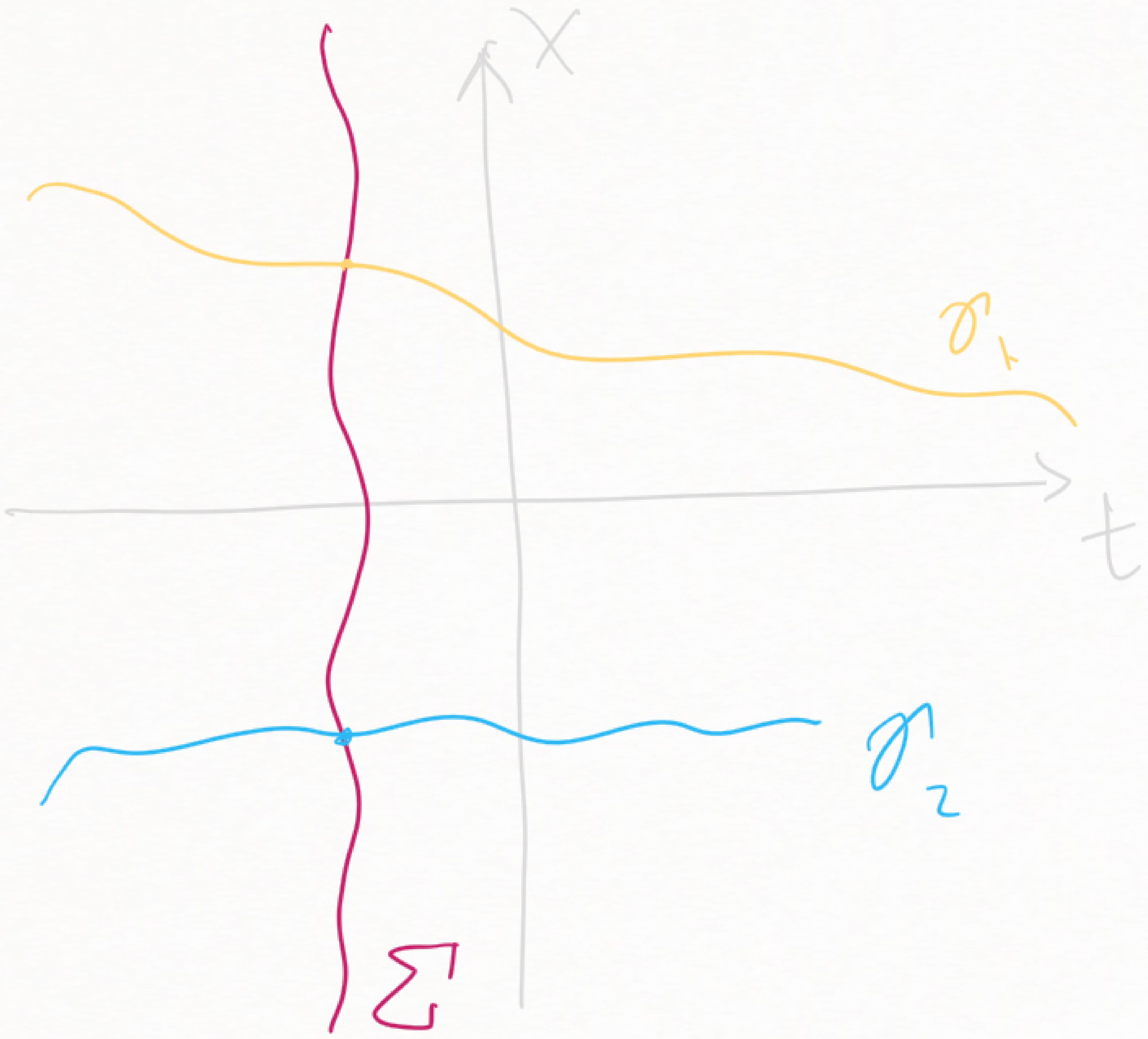
A curve is called inextendible if it has no extensions.

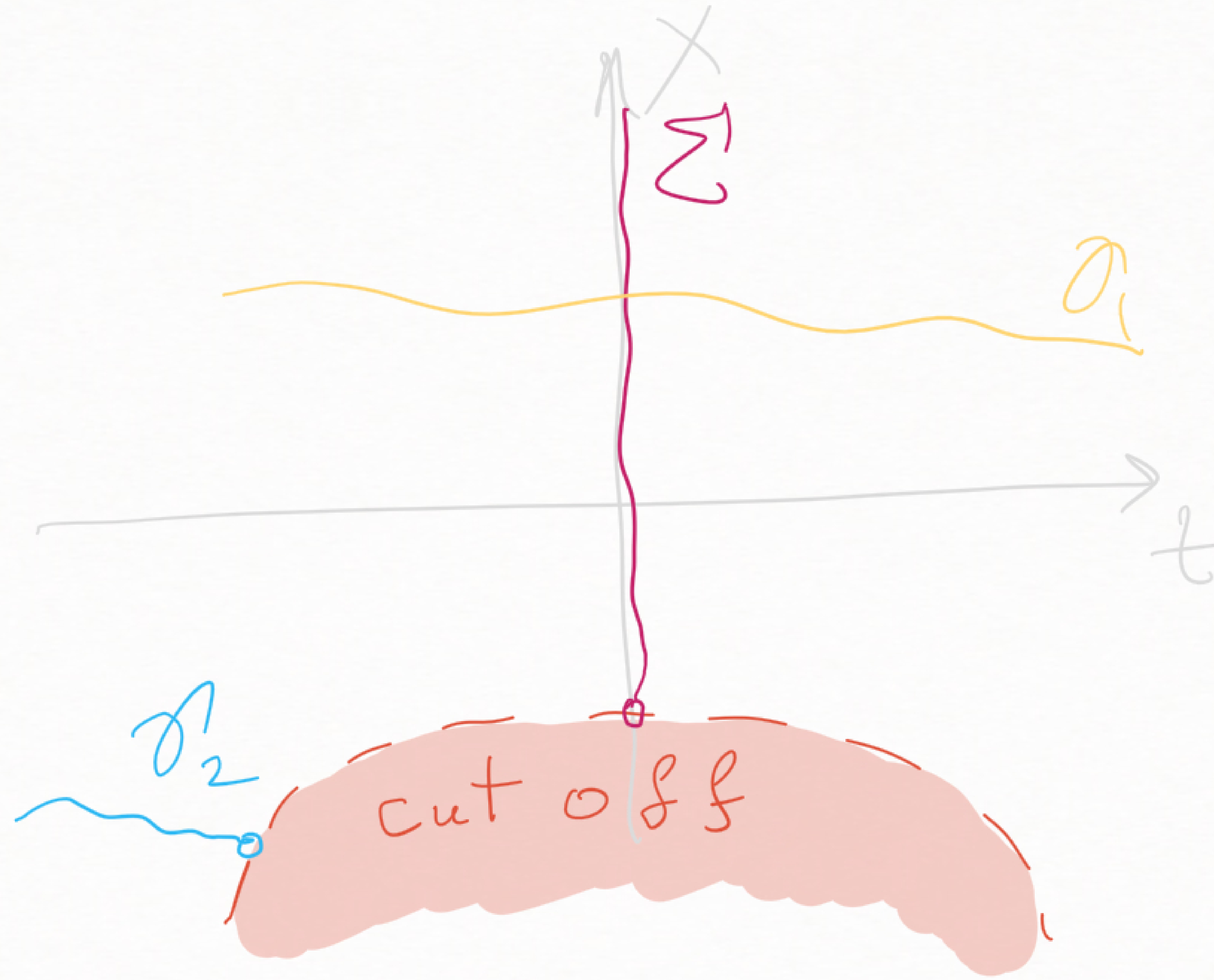
Global Hyperbolicity

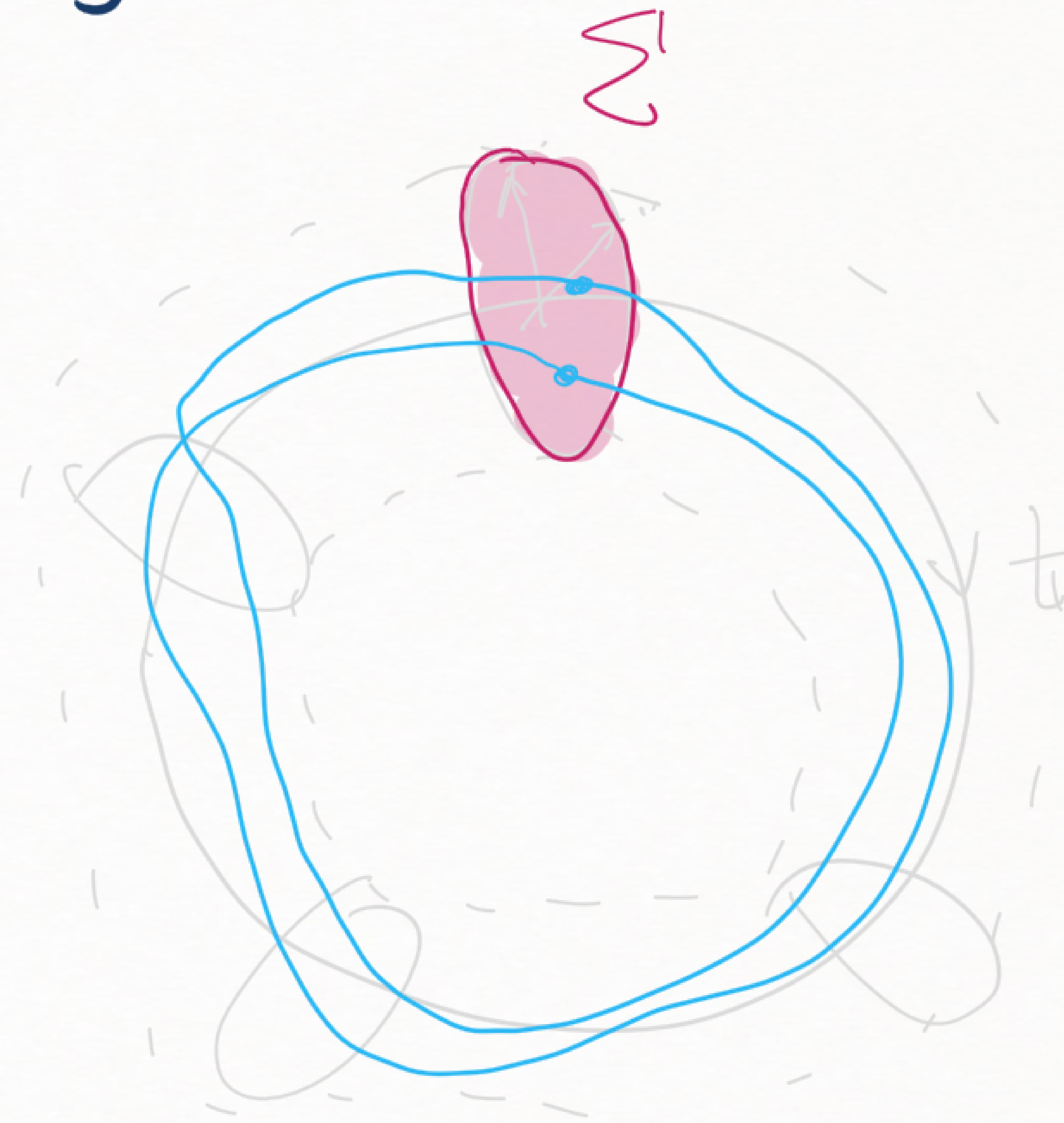
Cauchy hypersurface A C^0-hypersurface $\Sigma \subseteq M$ s.t.
$\forall\, \gamma \in C^1((\alpha,\beta), M)$ causal inextendible, $\#\gamma((\alpha,\beta)) \cap \Sigma = 1$, or

$$\left(\exists!\, s \in (\alpha,\beta)\right)\ \gamma(s) \in \Sigma.$$

(M,g) is **globally hyperbolic** if it has a Cauchy hypersurface







Global Time Function

Time function $t \in C(M)$ s.t. $\forall \gamma \in C^1([0,1], M)$ timelike, future-directed,
$(\forall s_1, s_2 \in [0,1])$ $s_1 < s_2 \Rightarrow t(\gamma(s_1)) < t(\gamma(s_2))$.

Early results: (M,g) glob. hyp. $\Rightarrow M \overset{\text{homeo}}{\simeq} \underline{\mathbb{R} \times \Sigma}$
$\Rightarrow \exists$ <u>global time function</u>

Questions:
- Is $\underline{M \overset{\text{diffeo}}{\simeq} \mathbb{R} \times \Sigma}$?
- Does exist C^∞ time function t, s.t.
  $\underline{M \xrightarrow{t} (\alpha, \beta)}$ trivial fibre bundle
  with Cauchy hypersurface fibres?

Factorization

Bernal, Sanchez 2003, 2005:

Theorem: Let (M, g) be globally hyperbolic. Then it is isometrically diffeomorphic to (with time orientation preserved)

$$\left(\mathbb{R}\times\Sigma, \beta^2 \oplus -h_*\right)$$

where $\beta \in C^\infty(\mathbb{R}\times\Sigma, (0,+\infty))$ and $\mathbb{R} \ni t \longmapsto h_t \in C^\infty(T_*\Sigma^{\otimes 2})$ is a smooth family of Riemannian metrics on Σ. Moreover, $(\forall t \in \mathbb{R})$ $\{t\}\times\Sigma = \Sigma_t \subseteq \mathbb{R}\times\Sigma$ is a Cauchy hypersurface.

$$g(t,x) = \beta^2(t,x) \oplus - h_t(x)$$

Classification

Which spacetimes $(\mathbb{R}\times\Sigma, \beta^2\oplus -h_*)$ are globally hyperbolic?

Classical: $\beta(t,x)=\beta(t)$, $h_t(x)=a(t)^2\cdot h(x)$, (Σ,h) complete

Choquet-Bruhat, Cotsakis 2002

1. $0<m\leq\beta(t,x)\leq M$
2. (Σ,h_0) complete
3. $\inf\limits_{X\in TM}\dfrac{h_t(X,X)}{h_0(X,X)}\geq A>0$

$\Bigg\}\Longrightarrow (\mathbb{R}\times\Sigma,\beta^2\oplus -h_*)$ globally hyperbolic

Does not cover homogeneous cosmological spacetimes or open causal diamonds $\mathcal{I}^+(p)\cap\mathcal{I}^-(q)$.

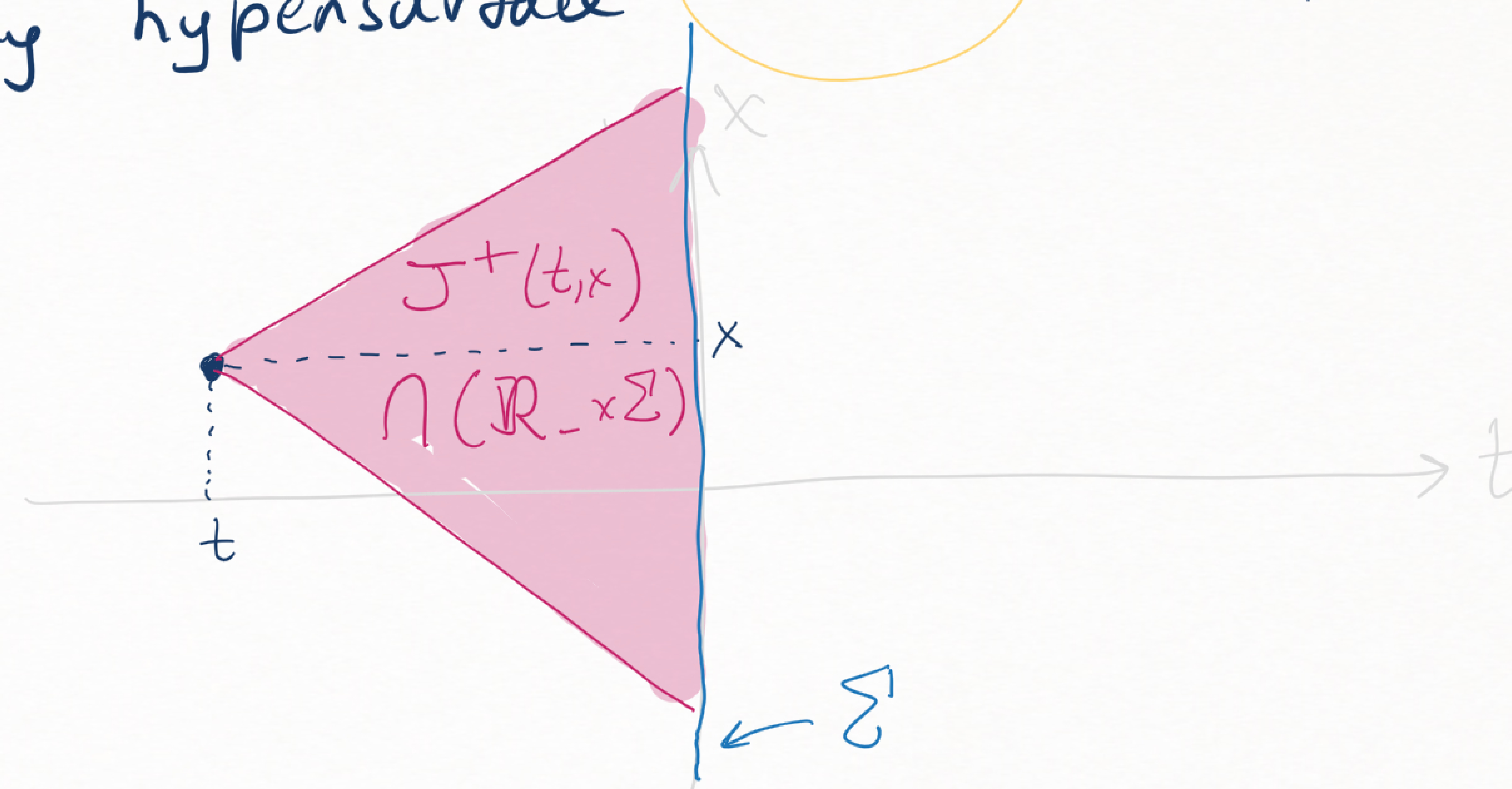
Classification (cont.)

Z.A. 2021: Let h_∞ be a C^0 Riemannian metric on Σ' s.t. (Σ', h_∞) is complete. Introduce $D : \mathbb{R}\times\Sigma' \longrightarrow \mathbb{R}_+$ as

$$D(t,x) = \frac{\beta(t,x)^2}{\min\limits_{\substack{X\in T_x\Sigma' \\ h_\infty[x](X,X)=1}} h_t[x](X,X)}, \qquad \forall (t,x)\in\mathbb{R}\times\Sigma'.$$

Then $\Sigma_0 = \{0\}\times\Sigma'$ is a Cauchy hypersurface $\Longleftrightarrow$ $\forall (t,x)\in\mathbb{R}\times\Sigma'$,

$$\sup_{\substack{J^+(t,x)\cap(\mathbb{R}_-\times\Sigma') \\ J^-(t,x)\cap(\mathbb{R}_+\times\Sigma')}} D < \infty.$$



Homogeneous Cosmological Spacetimes (cont.)

Let (G', M, g) be a homogeneous cosmological spacetime.

2 foliations: by G'-orbits v.s. by Cauchy hypersurfaces

Questions: Can the 2 foliations coincide? Minimal set of assumptions?

Z. A. 2021:

1. G' acts on M properly
2. Generators $\{X\}$ of G spacelike
   $\max\dim\{X\} = d$
3. G'-orbits connected

$$\Longrightarrow \quad (G', M, g) \simeq \left(\mathbb{R} \times (G', \Sigma'),\ \beta^2(t) - h_t\right),$$

$\Sigma_t = \{t\} \times \Sigma$ Cauchy hypersurface.

Homogeneous Cosmological Spacetimes (cont.)

Are conditions 1.–3. sharp/necessary/minimal?

Z. A. 2021: For connected G, compact $H \subseteq G$, $\Sigma = G/H$, and any spacetime $(\mathbb{R} \times (G, \Sigma), \beta^2 t - h_*)$, the conditions 1.–3. hold true.

Conclusion: Conditions 1.–3. are the appropriate definition of a homogeneous cosmological spacetime.

The equivariant factorization $M \simeq \mathbb{R} \times \Sigma$ is automatic.

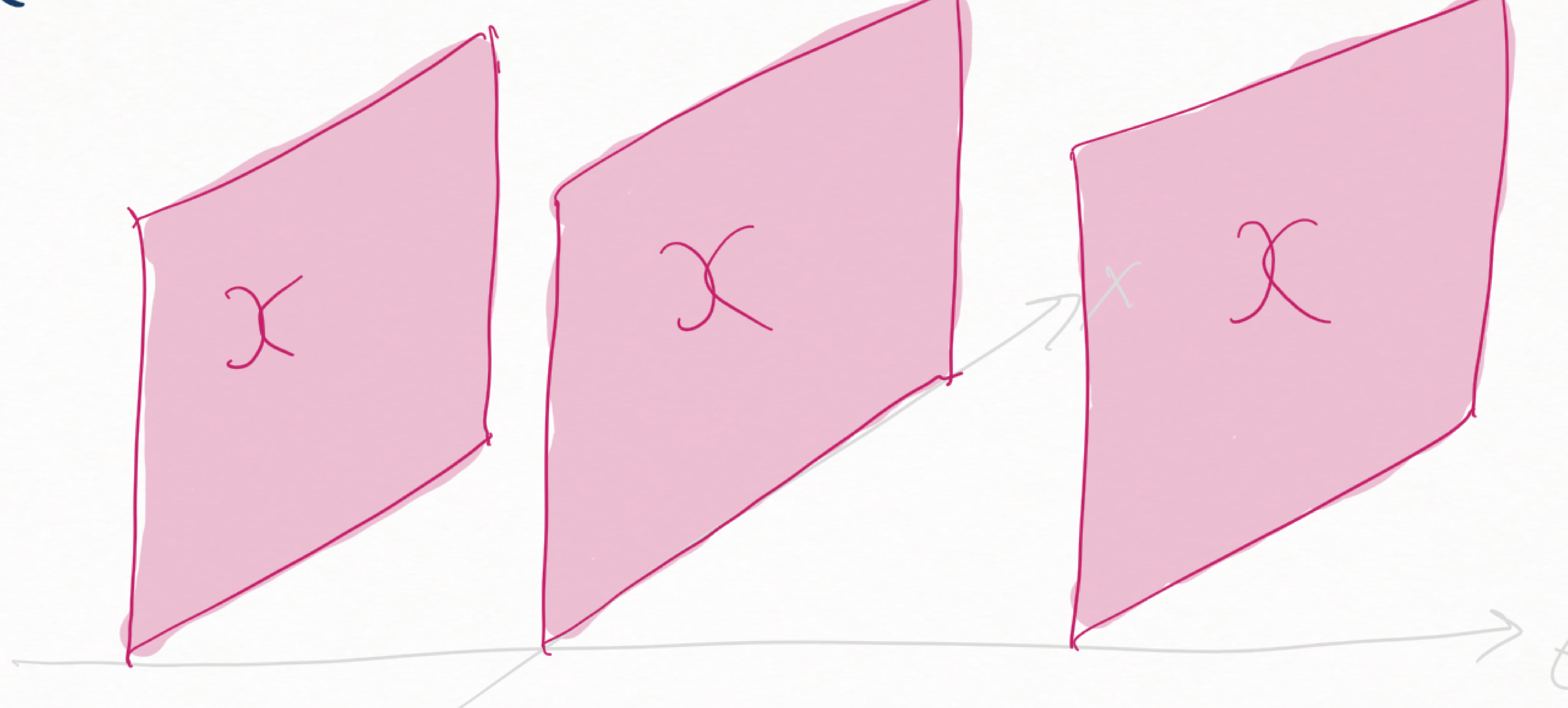
Homogeneous Cosmological Vector Bundles

Take $M = \mathbb{R} \times \Sigma$ and a vector bundle $\mathcal{J} \to \mathbb{R} \times \Sigma$.

Denote $\mathcal{X} = \mathcal{J}|_{\Sigma_0}$.

Question: $\mathcal{J} \simeq \mathbb{R} \times \mathcal{X}$?

Answer: Yes (standard differential topology)



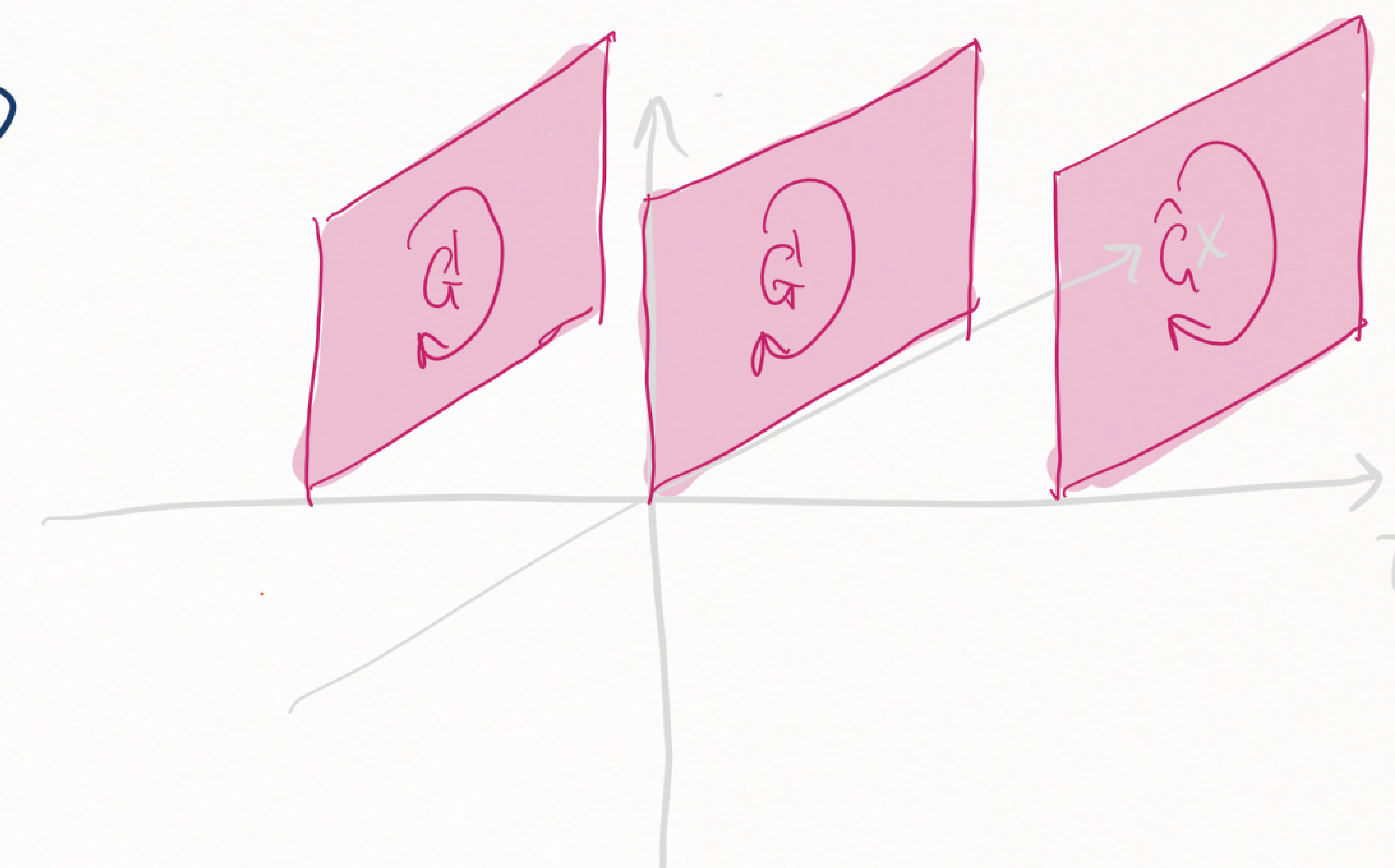
Equivariant Factorization

Homogeneous cosmological vector bundle

1. $(G, \mathbb{R}\times\Sigma) \simeq \mathbb{R}\times(G,\Sigma)$ hom. cosm. spacetime
2. G acts smoothly on the vector bundle
   $\mathcal{J} \xrightarrow{\pi} \mathbb{R}\times\Sigma$ s.t. π is G-equivariant.

Question: $(G,\mathcal{J}) \simeq \mathbb{R}\times(G,X)$?

Z.A. 2021: YES.



References

Bernal, Sanchez 2003 "On smooth Cauchy hypersurfaces and Geroch's splitting theorem", Comm. Math. Phys., 243.

Bernal, Sanchez 2005 "Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes", Comm. Math. Phys., 257

Choquet-Bruhat, Cotsakis 2002 "Global hyperbolicity and completeness", J. Geom. Phys., 43.

Z. A. 2021 "Global hyperbolicity and factorization in cosmological models", J. Math. Phys., 62(3).

Part II: Hyperbolic PDEs

Scalar PDEs

M C^∞ manifold, $P: C^\infty(M) \longrightarrow C^\infty(M)$ linear PDO with C^∞ coefficients ($\mathbb{R}$ or $\mathbb{C}$), of order $m \in \mathbb{N}_0$.

Local: $Pu(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u(x)$, $\quad a_\alpha \in C^\infty$

Invariant: $PDO_0(M) = C^\infty(M) \qquad Pu(x) = P(x)u(x), \quad \forall P \in PDO_0(M)$

$PDO_m(M) = \{P \in Hom(C^\infty(M), C^\infty(M)) \mid [P,Q] \in PDO_{m-1}(M), \forall Q \in PDO_0(M)\}$

$[\partial^m, a]u = \partial^m(au) - a\partial^m u = \partial a \cdot \partial^{m-1} u + \dots + \partial^m a \cdot u$

Peetre's theorem: $PDO(M) = \{P \in \mathcal{L}(C^\infty(M)) \mid \operatorname{supp} Pu \subseteq \operatorname{supp} u,\ \forall u \in C^\infty(M)\}$

Principal symbol: $P \in PDO_m(M)$ $\quad p_m \in C^\infty(T_*M)$ $\quad$ (not sections! $C^\infty(T_*M^{\otimes m} \to M)$)

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \qquad p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha \qquad (x,\xi) \in T_*M$$

$$p_m(q) = \frac{1}{m!} [\underbrace{Q,[Q,\dots,[Q}_{m},P]\dots](\pi(q)),$$

$Q \in PDO_0(M)$, $dQ \circ \pi(q) = q \in T_*M$

Elliptic

P elliptic at $x \in M$ if $p_m(x,\xi)$ an elliptic polynomial,

$(\forall \xi \in T_xM^*)$ $\quad p_m(x,\xi) = 0 \Rightarrow \xi = 0$

P elliptic on M if elliptic $\forall x \in M$

P elliptic if $\exists P^{-1} \in \Psi DO_{-m}(M)$

Hyperbolic

 No such thing as "hyperbolic per se".

Hyperbolic P is called hyperbolic at $x \in M$ w.r.t. $N \in T_x M^*$ "time" direction

if

- $P_m(x, N) \neq 0$
- $(\exists A > 0)(\forall \xi \in T_x M^*)(\forall t \in \mathbb{C})\ \mathrm{Im}\, t > A \Rightarrow P_m(x, \xi + tN) \neq 0$

This implies that $(\forall \xi \in T_x M^*)$ $P_m(x, \xi + tN) = (t - t_1) \cdots (t - t_m)$, $t_1, \ldots, t_m \in \mathbb{R}$.

If $t_1 < t_2 < \cdots < t_m$ then strongly/strictly hyperbolic.

Example

$$p_m(x,\xi) = \xi_1^2 + \xi_2^2 - \xi_3^2$$

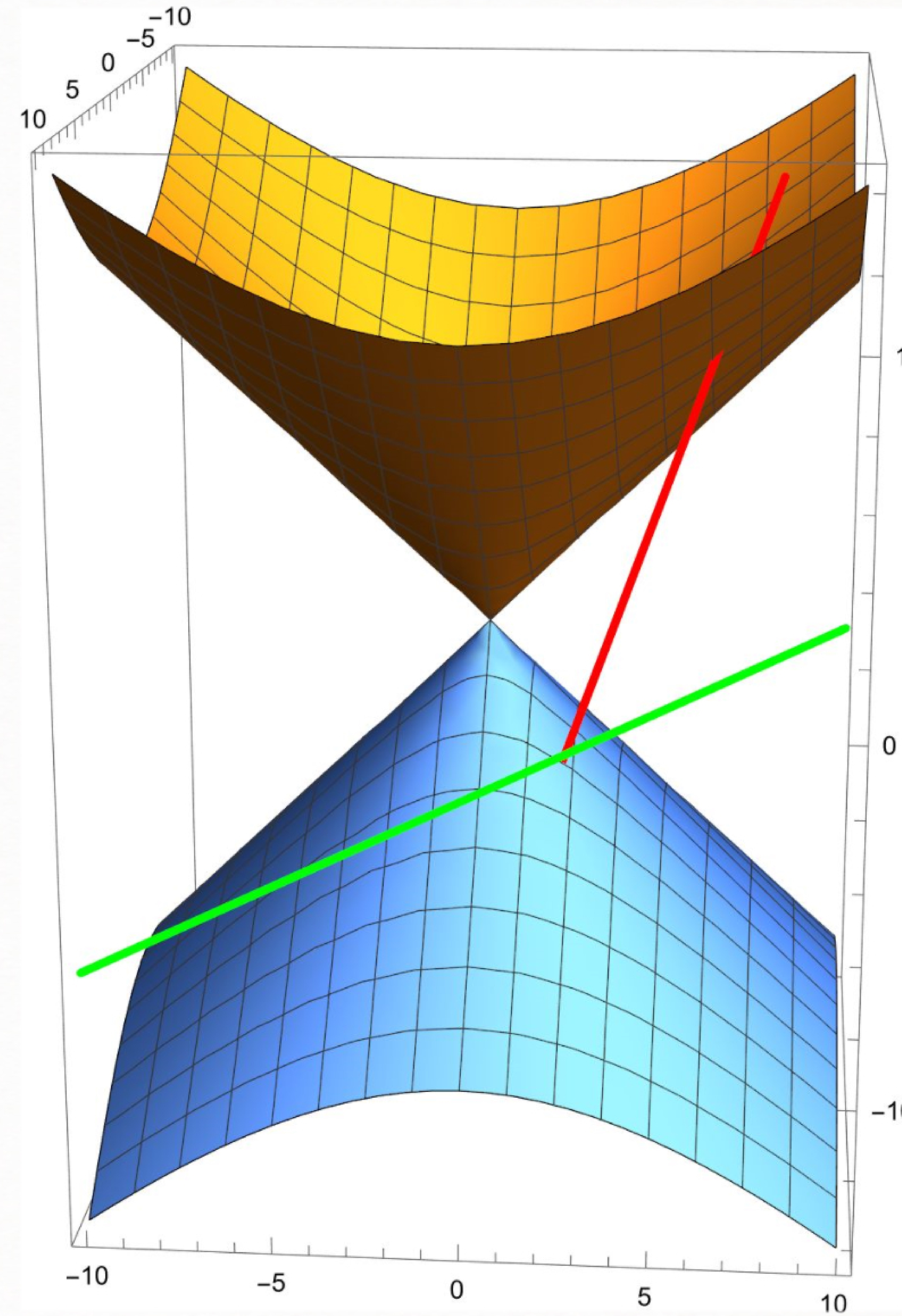
$N_1 = (1, 1, 2)$

$N_2 = (4, 2, 1)$

$$P = -\partial_{x_1}^2 - \partial_{x_2}^2 + \partial_{x_3}^2$$

Hyperbolic w.r.t. N_1

not hyperbolic w.r.t. N_2.



Hyperbolic (cont.)

It makes sense to speak of hyperbolicity on M w.r.t. a 1-form N on M.

On a spacetime (M,g)

τ time orientation, $N_\tau = g(\tau, \cdot)$ time direction

Normally hyperbolic $m=2$, $p_m(x,\xi) = g(x)(\xi,\xi)$, $(x,\xi) \in T_*M$.

Normally hyperbolic $\Rightarrow$ strictly hyperbolic w.r.t. N_τ.

Wave operators $\Box_g + X + c$

Cauchy-hyperbolic

M C^∞ manifold, $P \in PDO_m(M)$, $\Sigma \subseteq M$ C^∞ hypersurface

$N \in C^\infty(T_*M|_\Sigma)$, $N|_{T\Sigma} = 0$ (normal covector)

Cauchy problem (global)

$$\begin{cases} Pu = f \in C^\infty(M) \\ B^{m-1} u|_\Sigma = u_{m-1} \in C^\infty(\Sigma) \\ \vdots \\ B^0 u|_\Sigma = u_0 \in C^\infty(\Sigma) \end{cases}$$

$\Rightarrow WF(u) \subseteq Char(P) \cup WF(f)$

$WF(f) = \emptyset$

but we want $u \in C^\infty(M)$

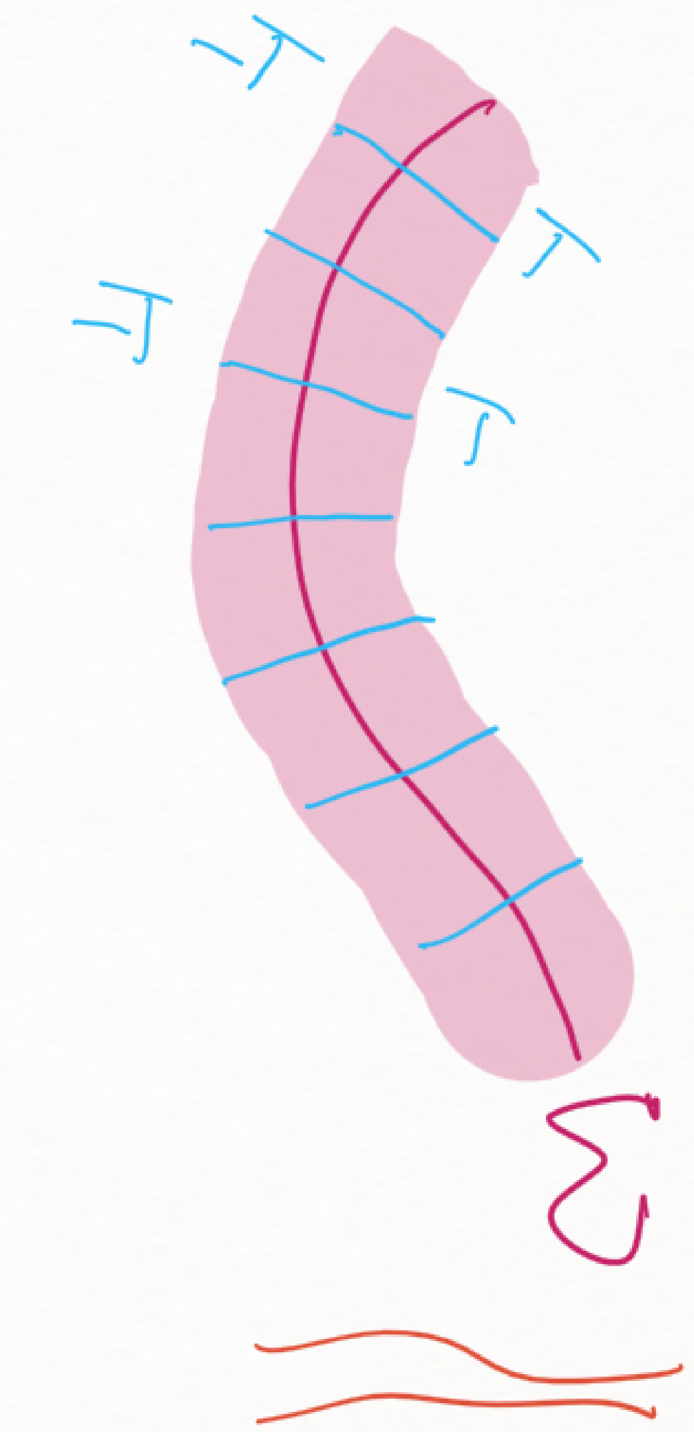
$B^j \in PDO_j(M)$, $j = 0, \dots, m-1$

$b_j(N) \neq 0$.

Cauchy-hyperbolic (cont.)

$\Sigma \subseteq U \subseteq M$, U open, $U \overset{\psi}{\simeq} [-T, T] \times \Sigma \ni (t, x)$

$d\psi_* N = dt$



Cauchy problem (local)

$$
\begin{cases}
Pu = f \in C^\infty(U) \\
\partial_t^{m-1} u|_{t=0} = u_{m-1} \in C^\infty(\Sigma) \\
\vdots \\
u|_{t=0} = u_0 \in C^\infty(\Sigma)
\end{cases}
$$

Theorem:

P strictly hyp. w.r.t. N in U $\Longrightarrow$ Well-posed, $\exists!\, u$

Hyperbolic vs Cauchy-hyperbolic

(strictly, or analytic etc.) hyperbolic $\rightsquigarrow$ (local etc.) Cauchy-hyperbolic

Conversely

Theorem: Cauchy problem "H_∞-well-posed" in $[-T,T] \times \{$

$$\Downarrow$$

$$\underline{P_m(0, x; sN) = (s-s_1) \cdots (s-s_m) \cdot \mathrm{const}}$$

P hyperbolic w.r.t. N at $t=0$

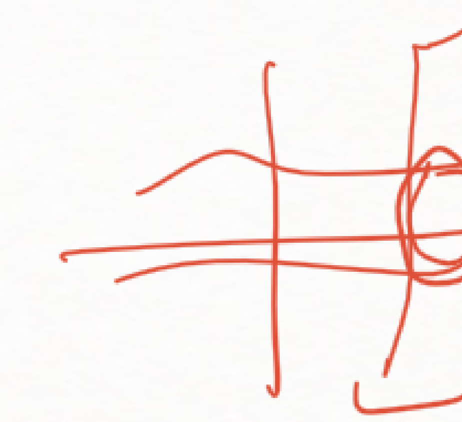
Causally Cauchy-hyperbolic

(M,g) spacetime, τ time-orientation, $\Sigma \subseteq M$ spacelike hypersurface, $g(\tau, T\Sigma) = 0$, $N = g(\tau, \cdot)$.

$\tau \sim \partial_t$

Cauchy problem (global)

spatially compactly supported



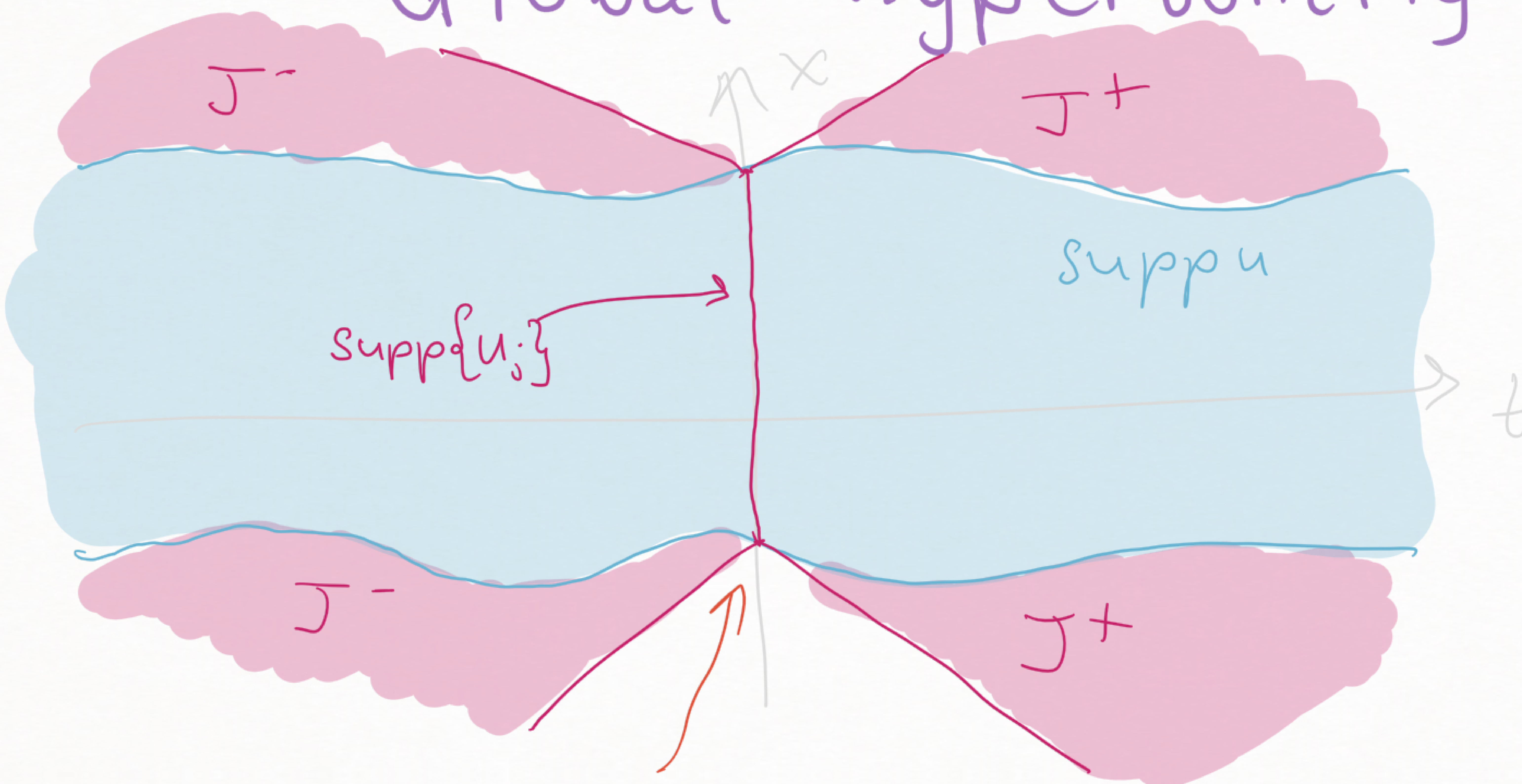
$$
\begin{cases}
Pu = 0 \\
\tau^{m-1} u\big|_{\Sigma} = u_{m-1} \in C^\infty_c(\Sigma) \\
\vdots \\
u\big|_{\Sigma} = u_0 \in C^\infty_c(\Sigma)
\end{cases}
$$

Causal well-posedness

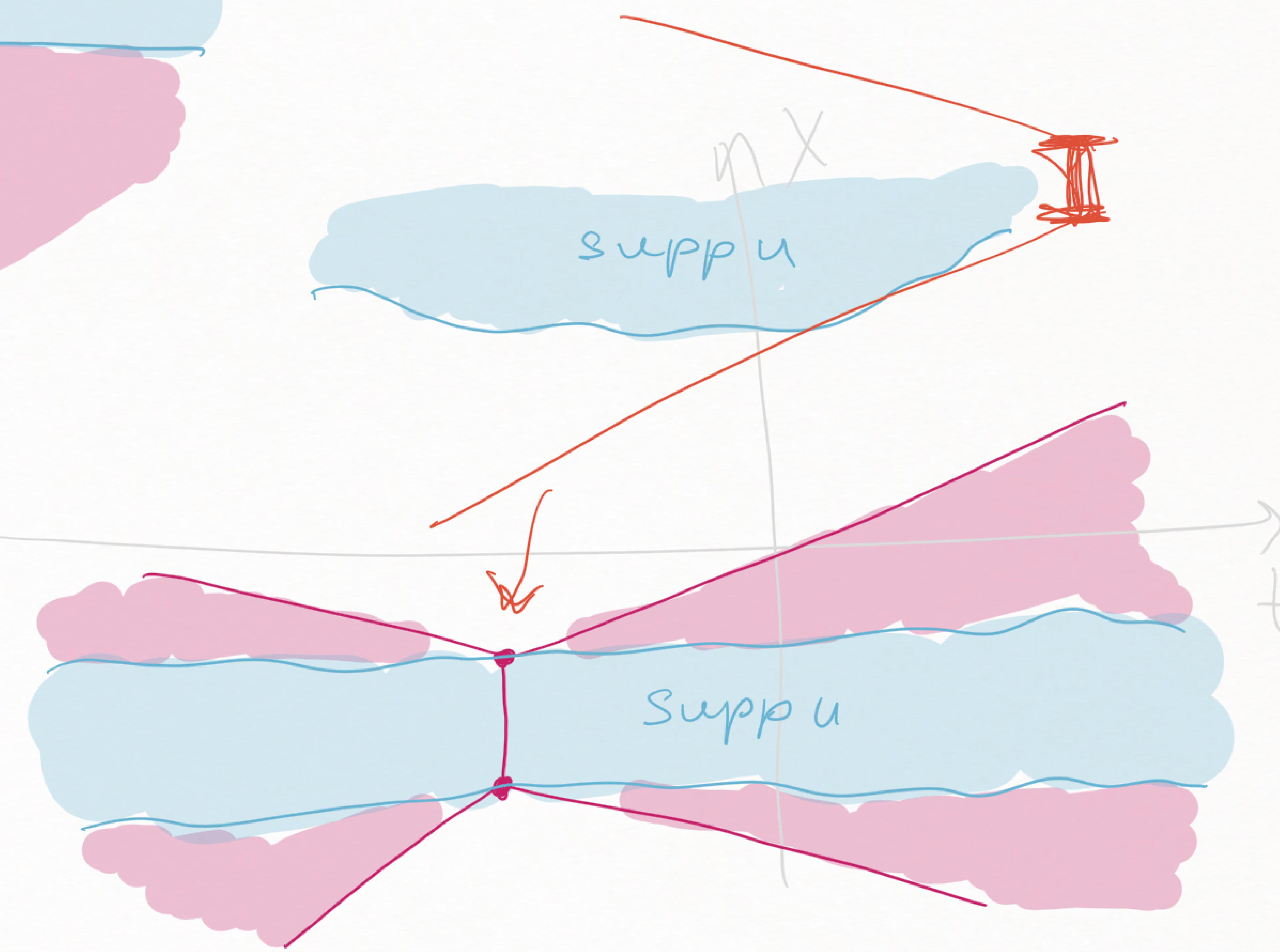
$$\{u_j\}_{j=0}^{m-1} \xmapsto{\ S\ } u \in C^\infty_{sc}(M)$$

$$\operatorname{supp} u \in \bigcup_{j=0}^{m-1} J^+(\operatorname{supp} u_j) \cup J^-(\operatorname{supp} u_j)$$

Global hyperbolicity (is necessary)



Good



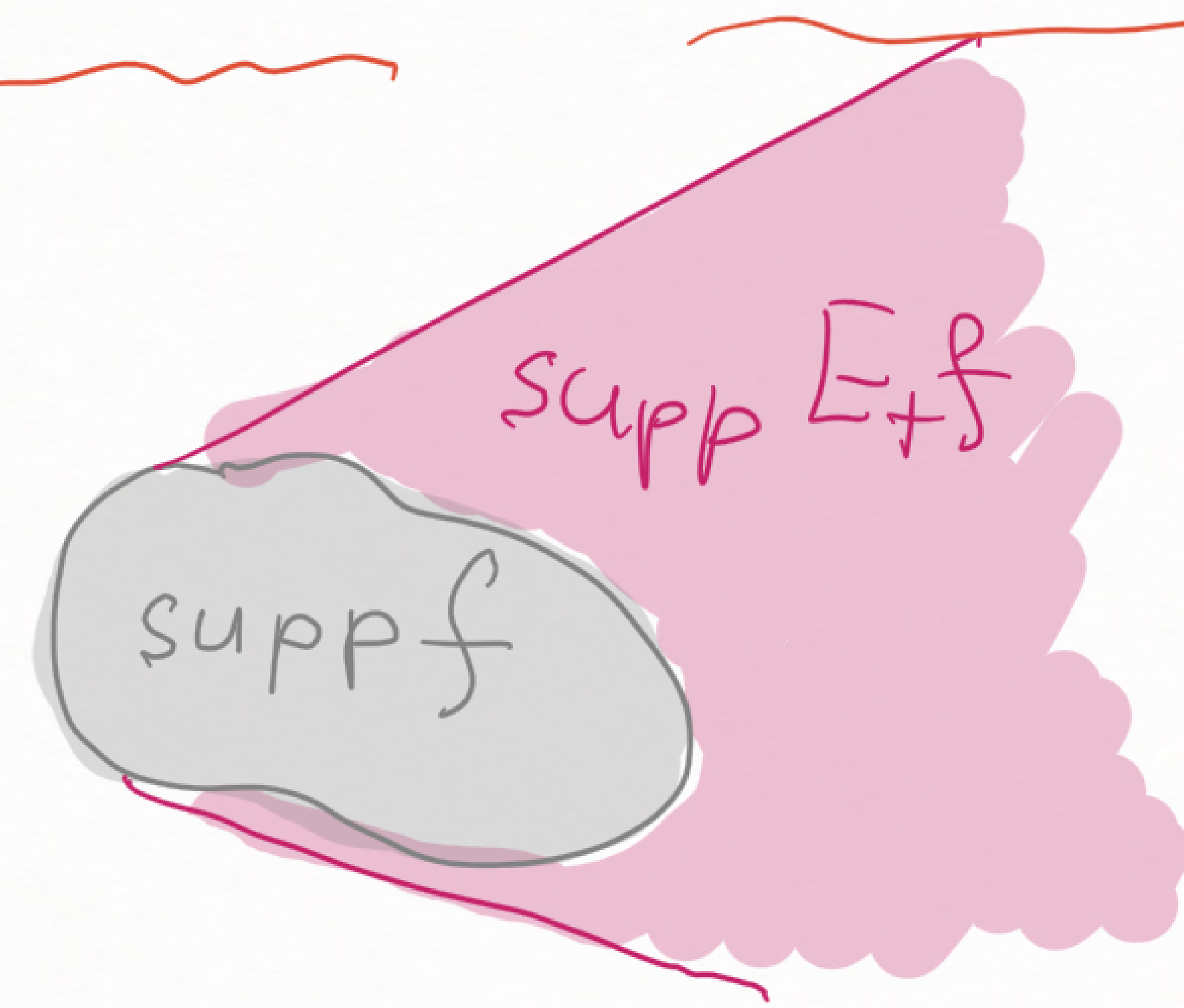
Bad

Green-hyperbolic

(M,g) spacetime, τ time-orientation, $P \in PDO_m(M)$

Advanced/retarded Green's functions

- $E_\pm : C_c^\infty(M) \longrightarrow C_{sc}^\infty(M)$
- $PE_\pm = \mathbb{1}_{C_c^\infty(M)}$ • $E_\pm P = \mathbb{1}_{C_{sc}^\infty(M)}$
- $\mathrm{supp}(E_\pm f) \subseteq J^\pm(\mathrm{supp} f)$, $\forall f \in C_c^\infty(M)$



$Pu = f$, $f \in C_c^\infty(M)$

$\Downarrow$

$(\exists!\, u_+)$ $\mathrm{supp}\, u_+ \subseteq J^+(\mathrm{supp} f)$

$u_+ = E_+ f$

Example

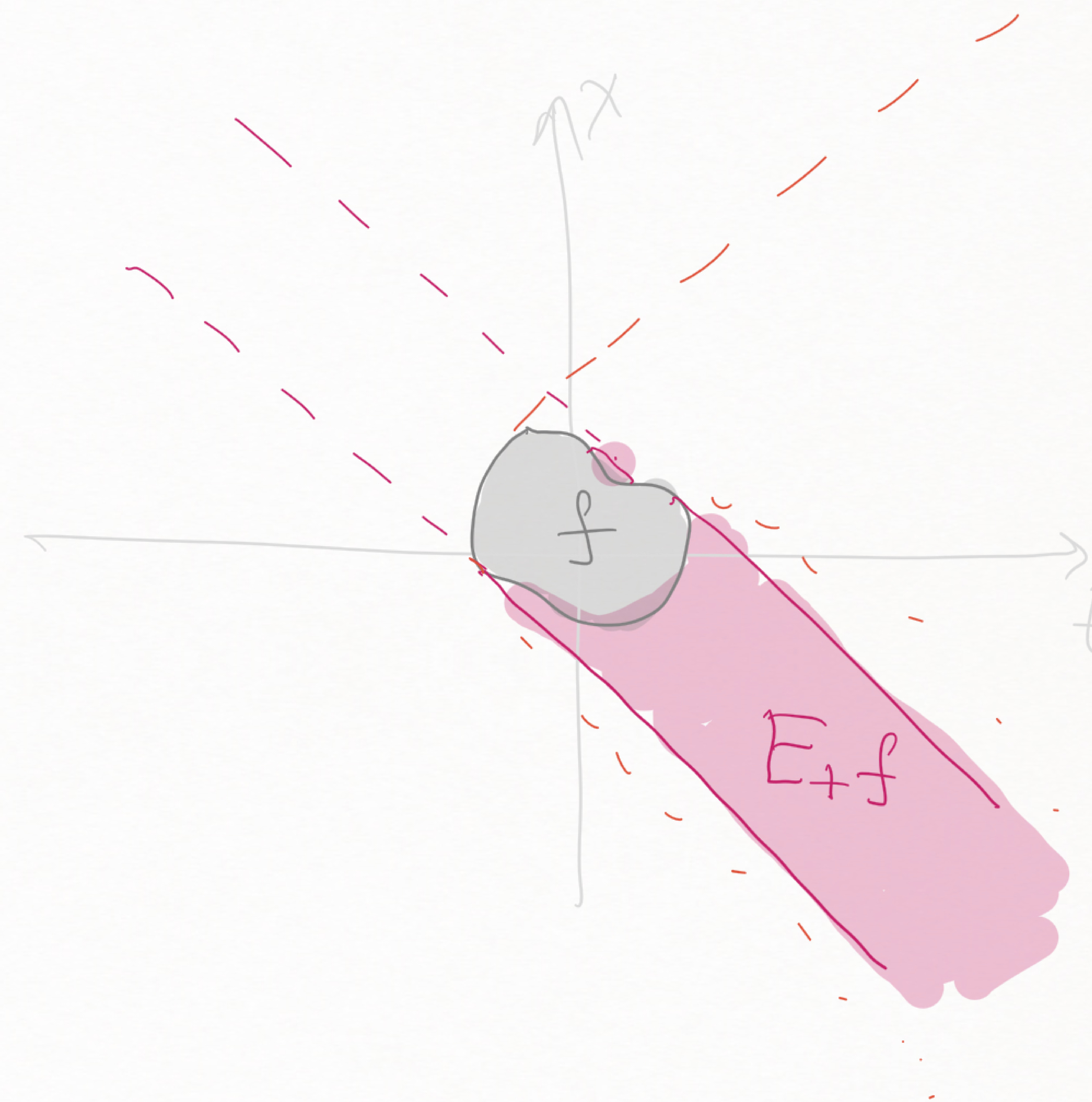
$Pu = f$

$(M, g) = \mathbb{R}^{1,1}$, $P = \partial_t - \partial_x$

$$E_+ f(t,x) = \frac{1}{2}\int_{-\infty}^{t-x} f\left(\frac{t+x+s}{2}, \frac{t+x-s}{2}\right) ds$$

$$E_- f(t,x) = -\frac{1}{2}\int_{t-x}^{+\infty} f\left(\frac{t+x+s}{2}, \frac{t+x-s}{2}\right) ds$$

$$u_t - u_x = f$$



Causal propagator

$E \triangleq E_+ - E_- : C_c^\infty(M) \to C_{sc}^\infty(M)$

$PE = EP = 0$

$$Ef(t,x) = \frac{1}{2}\int_{-\infty}^{\infty} f\left(\frac{t+x+s}{2}, \frac{t+x-s}{2}\right) ds$$

Example

$$E(t,x;s,y) = -2\,\delta\left(2[t+x-s-y]\right)$$

Fact: $E : C_c^\infty(M) \longrightarrow \{u \in C_{sc}^\infty(M) \mid Pu = 0\}$

surjective

Cauchy problem

$$u(0,x) = u_0(x) = E f_u(0,x) = \frac{1}{2}\int_{-\infty}^{\infty} f_u\left(\frac{x+s}{2}, \frac{x-s}{2}\right) ds$$

$$E f_u(t,x) = \frac{1}{2}\int_{-\infty}^{\infty} f_u\left(\frac{x+t+s}{2}, \frac{x+t-s}{2}\right) ds = E f_u(0, x+t)$$

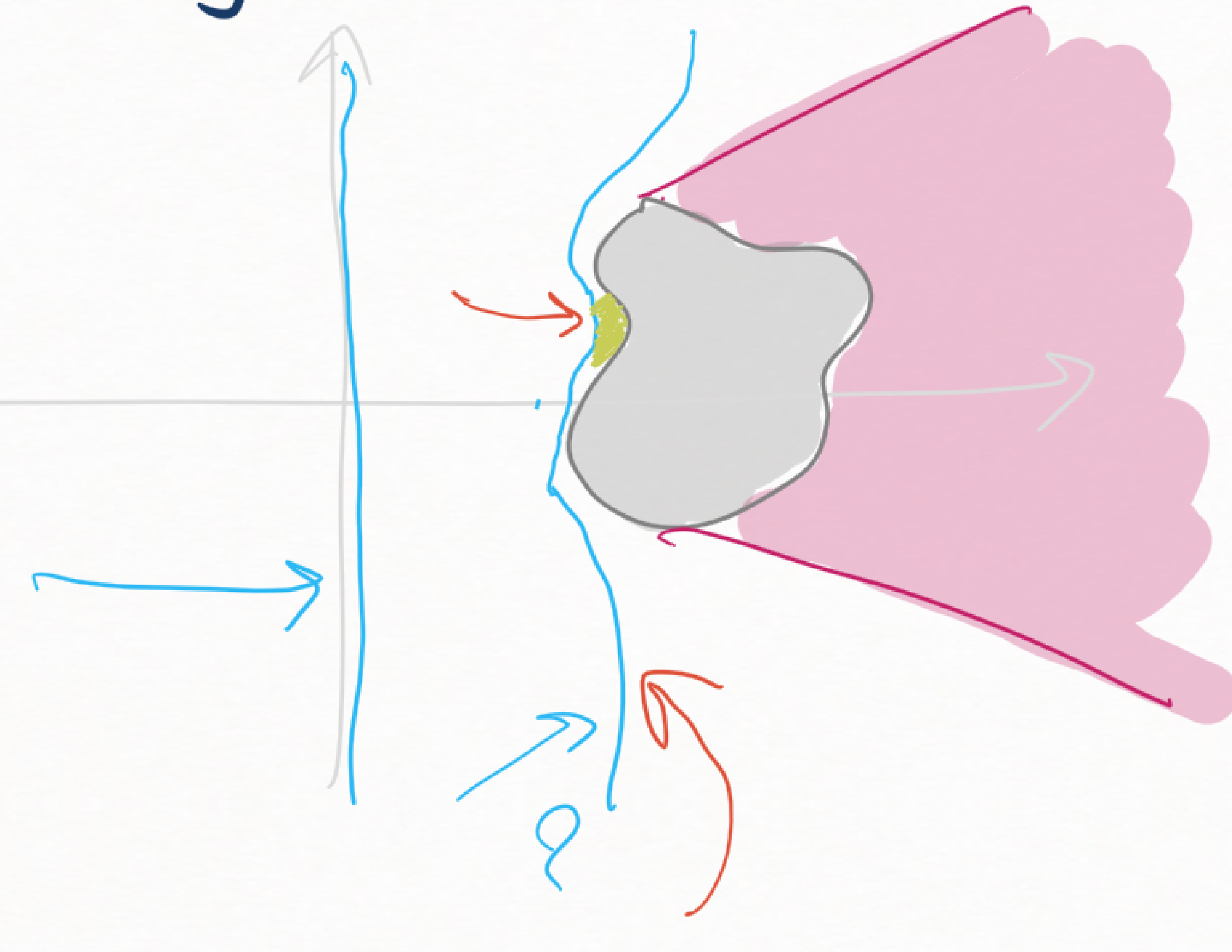
$$\boxed{u(t,x) = u_0(x+t)}$$

Green-hyperbolic $\overset{?}{\Longrightarrow}$ Cauchy-hyperbolic

Converse

$$\text{Cauchy-hyperbolic} \overset{?}{\Longrightarrow} \text{Green-hyperbolic}$$

$$\begin{cases} u_0 = 0 \\ \vdots \\ u_{m-1} = 0 \end{cases}$$

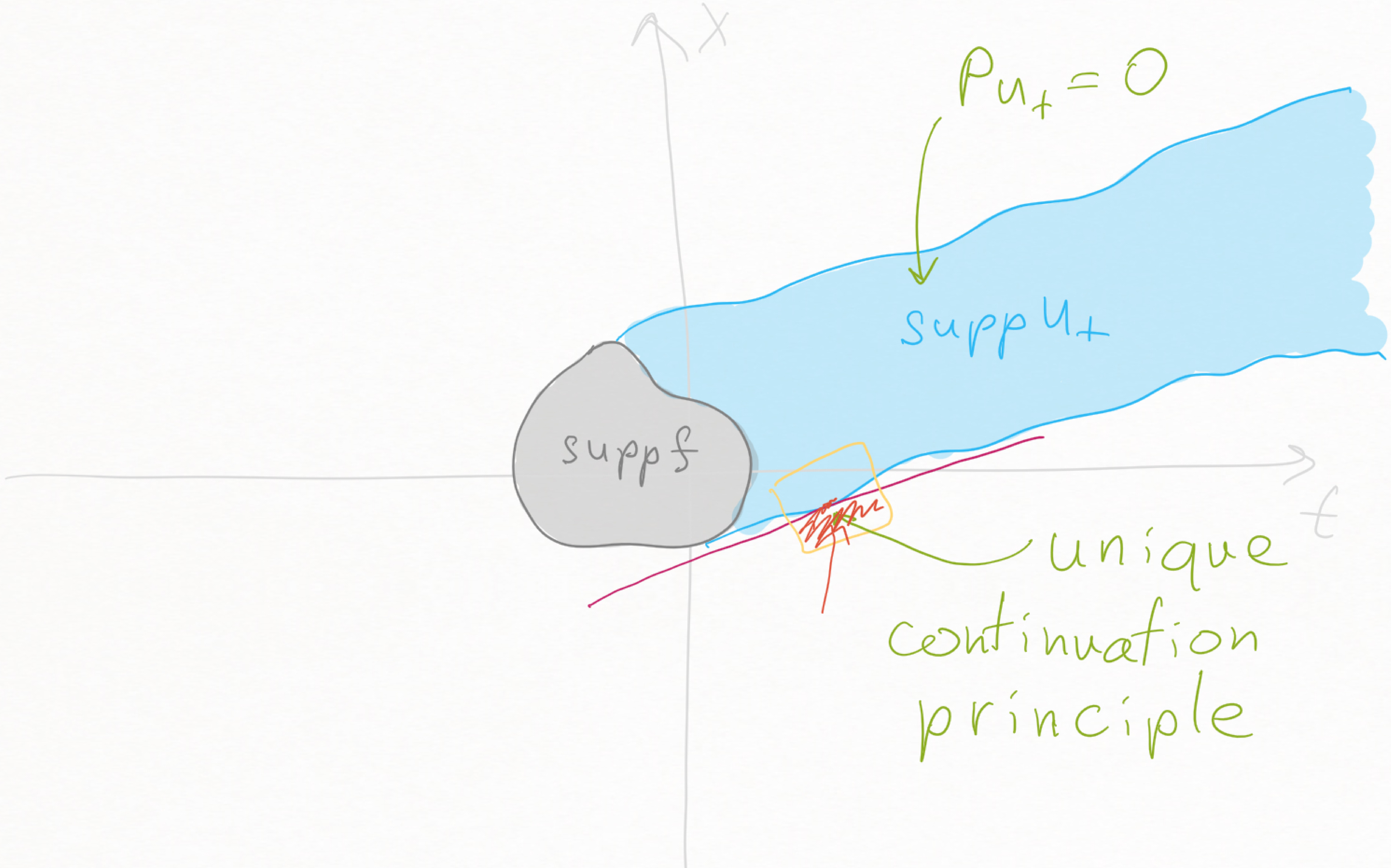


$$\begin{cases} Pu = f \in C_c^\infty(M) \\ \partial_t^{m-1} u|_\Sigma = 0 \\ \vdots \\ u|_\Sigma = 0 \end{cases}$$

$$\Downarrow$$

$$\exists!\, u \overset{?}{=} E_+ f$$

Green-hyperbolic vs hyperbolic



$Pu_+ = f$, $u_+ \in C^\infty_{sc}(M)$

$u_+ = E_+ f$ $\quad f \in C^\infty_c(M)$

Vector-valued PDEs

$\mathcal{J} \to M$ C^∞ vector bundle, $P : C^\infty(\mathcal{J}) \to C^\infty(\mathcal{J})$

linear PDO with C^∞ coefficients.

Local: $Pu(x) = \sum_{|\alpha| \leq m}' a_\alpha(x) D^\alpha u(x)$, $a_\alpha \in C^\infty(\mathrm{Hom}(\mathcal{J}))$

The invariant definition and Peetre's theorem remain valid.

Principal symbol: $P \in PDO_m(\mathcal{J})$, $p_m \in C^\infty(T^*M \times \mathrm{Hom}(\mathcal{J}))$

$$p_m(x, \xi) = \sum_{|\alpha| = m}' a_\alpha(x)(i\xi)^\alpha$$

Microlocal

Characteristic set $\mathrm{Char}(P) = \{(x,\xi,w) \mid p_m(x,\xi)w = 0\} \subseteq T_*M \oplus J$

Polarization set $WF_{pol}(u) = \bigcap\limits_{\substack{P: C^\infty(J) \to C^\infty(M) \\ Pu \in C^\infty(M)}} \mathrm{Char}(P)$

Elliptic P elliptic at $x \in M$ if $\det p_m(x,\xi)$ elliptic

Hyperbolic P (strictly) hyperbolic at $x \in M$

w.r.t. $N \in (T_x M)^*$ if $\det p_m(x,\xi)$ (strictly) hyperbolic.

strictly hyperbolic $\Rightarrow$ WF_{pol} propagates along Hamiltonian trajectories

Causal Cauchy problem

$(M,g)=(\mathbb{R}\times\Sigma, \beta^2 \oplus -h_t)$ globally hyperbolic spacetime, $N=\beta^2 dt$, $\tau=\partial_t$

Problem: What do we mean by $\partial_t u\big|_{t=0}$ for $u\in C^\infty(\mathcal{J})$?

$$\lim_{t\to 0}\frac{u(t,\cdot)-u(0,\cdot)}{t}$$

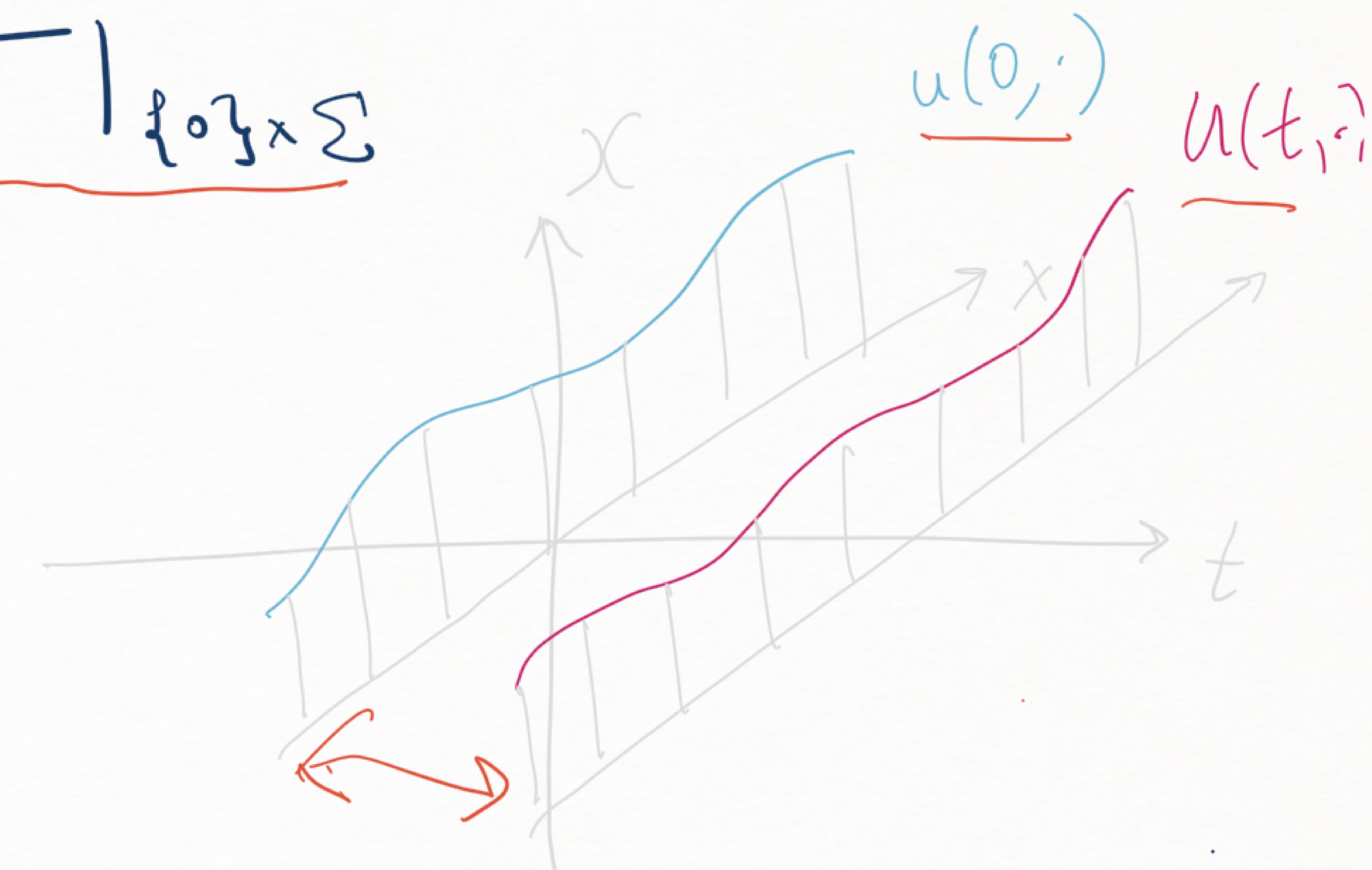
$u(t,x)\in \mathcal{J}\big|_{\{t\}\times\Sigma}$

$u(0,x)\in \mathcal{J}\big|_{\{0\}\times\Sigma}$

Factorization: $\mathcal{J}\simeq\mathbb{R}\times\mathcal{X}$, $\mathcal{X}=\mathcal{J}\big|_{\{0\}\times\Sigma}$

$$\mathbb{R}\ni t\xmapsto{u} u(t,\cdot)\in C^\infty(\mathcal{X})$$

$$u(t,x)\in\mathcal{J}\big|_{\{t\}\times\Sigma}\simeq\mathcal{J}\big|_{\{0\}\times\Sigma}=\mathcal{X}\quad\forall t\in\mathbb{R}$$



Causal Cauchy problem (cont.)

$P \in PDO_m(\underline{\mathbb{R}\times X})$

$$\begin{cases} Pu = f \in C_c^\infty(\mathbb{R}\times X) \\ \partial_t^{m-1} u\big|_{t=0} = u_{m-1} \in \underline{C_c^\infty(X)} \\ \vdots \\ u\big|_{t=0} = u_0 \in \underline{C_c^\infty(X)} \end{cases}$$

$\underline{\exists!}\; u \in C_{sc}^\infty(\mathbb{R}\times X)$

$\underline{\operatorname{supp} u} \subseteq \underline{J^+(\operatorname{supp}\{u_j\}_{j=0}^{m-1}) \cup J^-(\operatorname{supp}\{u_j\}_{j=0}^{m-1})}$
$\cup \underline{J^+(\operatorname{supp} f) \cup J^-(\operatorname{supp} f)}$

Normal hyperbolic $\quad \underline{p_m(x,\xi) = g(x)(\xi,\xi)\cdot \mathbb{1}}, \quad \forall (x,\xi) \in T_*M$

$m=2$

Theorem: $\underline{\text{Normal hyperbolic}}$ P are (globally) $\underline{\text{Cauchy-hyper}}$-bolic.

Hyperbolic vs Cauchy-hyperbolic

(strictly, or 1st order…) hyperbolic $\not\Longrightarrow$ (local, etc.) Cauchy-hyperbolic

Series of papers by Garetto, Jäh, Ruzhansky

Problem: global existence (long-time estimates) rely on energy functionals that involve fibre metric.

Factorization $P = \sum_{n=0}^{m_*} A_n(t)\partial_t^n \quad A_n(t) \in PDO_{m-n}(X)$

Converse

Cauchy-hyperbolic $\not\Rightarrow$ Hyperbolic

("H_∞-well-posed" + 1st order + regularity of spectrum...) (i.e., spectrum real at $t=0$...)

Higher order

$$[Op(p), Op(q)] = Op(\{p,q\}) \bmod PDO_{m-1}$$

Microlocal diagonalization not enough

$R = C^\infty(J)[\partial]$, $p \in \mathcal{M}(R, n)$ non-commutative linear algebra

Green-hyperbolic

Green's functions $E_\pm : C_c^\infty(\mathcal{J}) \longrightarrow C_{sc}^\infty(\mathcal{J})$, $Pu = f \in C_c^\infty(\mathcal{J})$

$$\exists! \, u_\pm = E_\pm f \in C_{sc}^\infty(\mathcal{J}) \quad \text{s.t.} \quad \operatorname{supp} u_\pm \subseteq J^\pm(\operatorname{supp} f)$$

The concept was introduced 2013-2015 by C. Bär.

Theorem: The class of Green-hyperbolic operators is closed under

- compositions
- spacetime embeddings
- direct sums

Example

$(M,g) = \mathbb{R}^{1,1}$ $\qquad \mathcal{J} = \mathbb{R}^{1,1} \times \mathbb{R}^2$

$$P = \begin{pmatrix} \partial_t - \partial_x & 0 \\ 0 & 1 \end{pmatrix} \qquad P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} \iff \begin{cases} u_t - u_x = f \\ v = h \end{cases}$$

$$E_\pm \begin{pmatrix} f \\ h \end{pmatrix} = \begin{pmatrix} E_\pm f \\ h \end{pmatrix} \qquad P_m(x,\xi) = \begin{pmatrix} i\xi_t - i\xi_x & 0 \\ 0 & 0 \end{pmatrix}$$

P neither hyperbolic

nor Cauchy-hyperbolic

Example 2

$$P = \begin{pmatrix} \partial_t - \partial_x & Q \\ 0 & 1 \end{pmatrix} \quad \forall Q \in PDO_m(\mathbb{R}^{1,1}), \quad m > 1.$$

$$P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} \iff \begin{cases} u_t - u_x + Qv = f \\ v = h \end{cases} \iff \begin{cases} u_t - u_x = f - Qh \\ v = h \end{cases}$$

$$E_\pm \begin{pmatrix} f \\ h \end{pmatrix} = \begin{pmatrix} E_\pm(f - Qh) \\ h \end{pmatrix} \qquad p_m(x, \xi) = \begin{pmatrix} 0 & q_m(x,\xi) \\ 0 & 0 \end{pmatrix}$$

The principal symbol doesn't cut it, and is sometimes marginally relevant.

Restricted Cauchy problem

$$P = \begin{pmatrix} \partial_t - \partial_x & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} u_t - u_x + v = 0 \\ v = 0 \end{cases} \qquad \begin{aligned} u(t,x) &= F(t+x) \\ v(t,x) &= 0 \end{aligned}$$

$$\begin{cases} P\begin{pmatrix} u \\ v \end{pmatrix} = 0 \\ \begin{pmatrix} u \\ v \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \end{cases}$$

$$u(0,x) = F(x) = u_0(x)$$

$$v(0,x) = 0 = v_0(x)$$

Restricted well-posedness

$$\begin{matrix} C_c^\infty(\mathbb{R}) \\ \oplus \\ 0 \end{matrix} \ni \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \longmapsto \begin{pmatrix} u \\ v \end{pmatrix} \in \begin{matrix} C_{sc}^\infty(\mathbb{R}^{1,1}) \\ \oplus \\ 0 \end{matrix}$$

Degeneracy vs Symmetry

$$\underbrace{\begin{pmatrix} \partial_t - \partial_x & Q \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}$$

Reducing the order

$$\underbrace{\begin{pmatrix} 1 & -Q \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} \partial_t - \partial_x & Q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \boxed{\begin{pmatrix} \partial_t - \partial_x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f - Qh \\ h \end{pmatrix}}$$

Reducing degeneracy

$$\begin{pmatrix} \partial_t - \partial_x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 0 \\ 0 & \partial_t - \partial_x \end{pmatrix}}\begin{pmatrix} \partial_t - \partial_x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \boxed{\begin{pmatrix} \partial_t - \partial_x & 0 \\ 0 & \partial_t - \partial_x \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ h_t - h_x \end{pmatrix}}$$

hyperbolic

$$\Updownarrow$$

$$\begin{pmatrix} \partial_t - \partial_x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \in \begin{pmatrix} f \\ h \end{pmatrix} + \underbrace{\mathrm{Ker}\begin{pmatrix} 1 & 0 \\ 0 & \partial_t - \partial_x \end{pmatrix}}$$

Algebraic quantization

(M,g) globally hyperbolic spacetime, $\mathcal{J} \to M$ vector bundle

$P: C^\infty(\mathcal{J}) \to C^\infty(\mathcal{J})$ Green-hyperbolic

$\mathrm{Sol}_{sc} \triangleq \mathrm{Ker}\, P \cap C^\infty_{sc}(\mathcal{J}) = \{u \in C^\infty_{sc}(\mathcal{J}) \mid Pu = 0\}$

$E = E_+ - E_- : C^\infty_c(\mathcal{J}) \to \mathrm{Sol}_{sc}(\mathcal{J})$ surjective

$\mathrm{Ker}\, E = P C^\infty_c(\mathcal{J})$.

$$C^\infty_c(\mathcal{J}) / P C^\infty_c(\mathcal{J}) \xrightarrow{E} \mathrm{Sol}_{sc}(\mathcal{J})$$

isomorphism

$$C^\infty_c(\mathcal{J}) \ni f \longmapsto [f]_P = f \bmod P C^\infty_c(\mathcal{J})$$

Isomorphism $\quad [f]_P \longmapsto Ef$

Algebraic quantization (cont.)

Symplectic form $\sigma([f],[h]) \triangleq \langle Ef, h\rangle_g$, $\langle\cdot,\cdot\rangle_g$ bilinear pairing

$\langle Pf, h\rangle_g = \langle f, Ph\rangle_g$, $\forall f,h \in C_c^\infty(\mathcal{J})$.

σ well-defined: $h-h' = Ph'' \Rightarrow \sigma([f],[h-h']) = \langle Ef, Ph''\rangle_g = \langle PEf, h''\rangle_g = 0$.

$\sigma(u,v) \triangleq \sigma([f_u],[f_v])$, $u,v \in Sol_{sc}$, $u = Ef_u$, $v = Ef_v$.

(Sol_{sc}, σ) - symplectic vector space

$*$-algebra $\mathcal{A} \triangleq \bigoplus_{n=0}^{\infty} [\mathbb{C} \otimes Sol_{sc}]^{\otimes n} / \mathcal{I}$, $\mathcal{I}$ ideal,

$A^* = \overline{A}$

$$\mathcal{I} = \langle u\otimes v - v\otimes u - \frac{i}{2}\sigma(u,v) \mid u,v \in Sol_{sc}\rangle$$

Algebraic quantization (cont.)

$\mathcal{A}$ - unital $*$-algebra, $\quad Sol_{sc} \overset{J}{\hookrightarrow} \mathcal{A}$ embedding.

"infinitesimal generators" of a C*-algebra

State ω, GNS rep. $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ s.t.

$(\forall A \in \mathcal{A})$ $\pi_\omega(A)$ densely defined on $\mathcal{H}_\omega$.

$$Sol_{sc} \ni u \longmapsto J(u) \longmapsto \pi_\omega(J(u)) \sim Op(u)$$