Fourier Analysis in Integrable Systems

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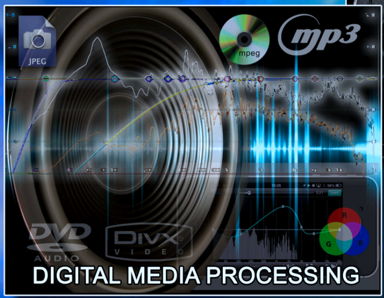
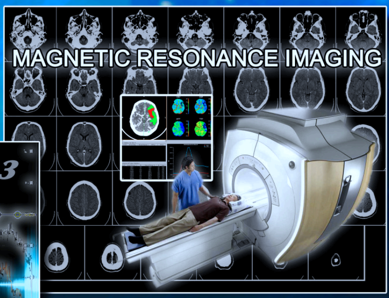
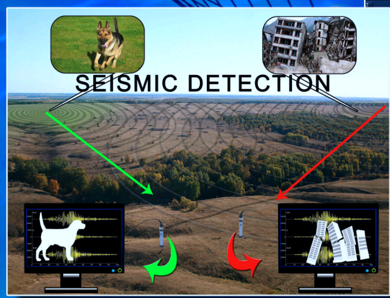
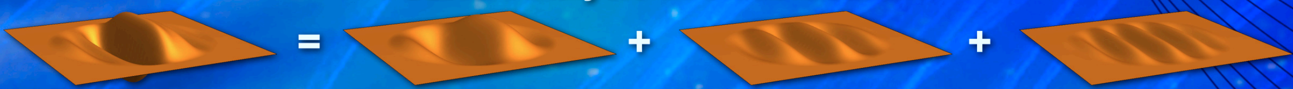






Fourier Analysis In a flat medium



Problem

Fourier transform can be defined as

$$\tilde{f}(k) = \mathcal{F}[f](k) = \int_M f(x)\bar{\zeta}_k(x)dx, \quad x \in M, k \in \tilde{M}, f \in L^2(M),$$

with ζ_k joint eigenfunctions of $\hat{p}_1, ..., \hat{p}_n$,

$$\hat{p}_i[\zeta_k](x) = \lambda_i(k)\zeta_k(x), \quad [\hat{p}_i, \hat{p}_j] = 0, \quad i, j = 1, ..., n, \quad \lambda_i(k) \in \mathbb{C}.$$

For classical (Euclidean) space

$$M = \mathbb{R}^n, \quad \tilde{M} = \mathbb{R}^n, \quad \hat{p}_i = -\imath\frac{\partial}{\partial x^i}, \quad \zeta_k(x) = e^{\imath\langle k,x\rangle}, \quad \lambda_i(k) = k_i.$$

Paley-Wiener properties (hold for Euclidean space):

- $\tilde{M}$ is the real form of a complex algebraic manifold $\tilde{M}^{\mathbb{C}}$
- $\lambda_i(k) \in \mathcal{H}(\tilde{M}^{\mathbb{C}})$ is a holomorphic function
- $\mathcal{F}[C_0^\infty(M)] = \left\{\tilde{f} \in \mathcal{H}(\tilde{M}^{\mathbb{C}}) : \quad |\tilde{f}(k+\imath l)| \leq C_N(1+\sum|\lambda_i(k)|)^{-N} e^{B\sum|\lambda_i(l)|}, \quad \forall N \in \mathbb{N}\right\}$
- $\mathcal{F}[C^\infty(M)'] = \left\{\tilde{f} \in \mathcal{H}(\tilde{M}^{\mathbb{C}}) : \quad |\tilde{f}(k+\imath l)| \leq C_N(1+\sum|\lambda_i(k)|)^{N} e^{B\sum|\lambda_i(l)|}\right\}$

Challenge: for an arbitrary integrable system,

- Can we solve for $\zeta_k(x)$ and $\lambda_i(k)$ explicitly?
- Can we prove (analogues of) Paley-Wiener properties?

Impact

- Cosmology: Gravitational waves, Particle creation, Microwave Background, ...
- Construction: Mechanical waves on solid surfaces
- Solid state physics: Qualitative behaviour in inhomogeneous media

Strategy

Functional Analysis

- $\tilde{M}$ is a topological space with Borel structure
- Fourier transform is unitary from $L^2(M)$ to $L^2(\tilde{M})$
- Explicit formulae are needed to establish Paley-Wiener properties

Algebraic Geometry

- Commutative rings of PDO studied by algebraic and complex analytic methods
- In many cases $\tilde{M}$ and ζ_k are found explicitly (see [2])
- Usually with M an algebraic variety and $\hat{p}_i$ having holomorphic coefficients
- Possibly also with $\hat{p}_i$ having real analytic coefficients

Harmonic Analysis

- In homogeneous spaces $\hat{p}_i$ do have real analytic coefficients
- Our Fourier transform is related to harmonic analytical Fourier transform
- Desired results already tangible for almost Abelian homogeneous spaces [1]

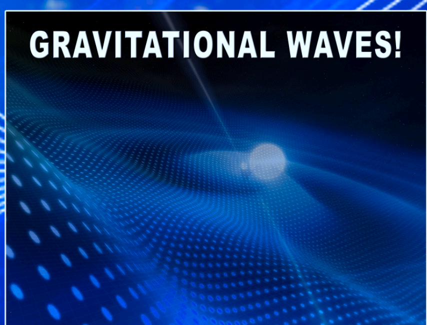
Integrable Systems

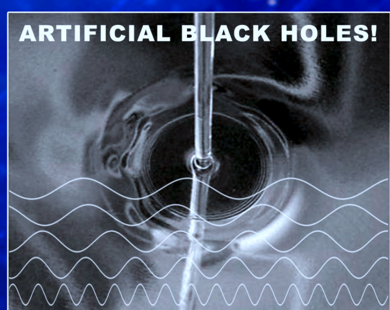
- Simplify systems using canonical transformations and Hamiltonian reduction

Outlook

- Fourier analysis in infinite dimensions: KdV, QFT, ...
- Harmonic and microlocal analysis: Quantization, Multipliers, ...

Fourier Analysis In a curved medium





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3. E. Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", Comm. Pure Appl. Math., vol. 13, No. I (1960)

The first author is supported by EPSRC