Mixed-Fourier-norm spaces and holomorphic functions

Zhirayr Avetisyan

Ghent Analysis and PDE Center at Ghent University, Belgium

Joint work with A. Karapetyants (Rostov-on-Don)

Seminar, 2 October 2025

Contents

- 1. Motivation
- 2. The General Scheme
- 3. Examples

Motivation

Motivation

The unit disc

- Unit disc $\mathbb{D} \subset \mathbb{C}$, polar coordinates $w = re^{2\pi\imath\varphi}$, $r \in (0, 1)$, $\varphi \in \mathbb{T}$.
- Angular Fourier transform

$$\mathfrak{F} f(n, r) = \hat{f}(n, r) = \int_{\mathbb{T}} f(\varphi, r) e^{-2\pi\imath n\varphi} d\varphi,$$

$$f(\varphi, r) = \sum_{n\in\mathbb{Z}} \hat{f}(n, r) e^{2\pi\imath n\varphi}.$$

Motivation

Cauchy-Riemann equations

- The Cauchy-Riemann operator in polar coordinates:

$$\frac{\partial}{\partial \bar{w}} = e^{2\pi\imath\varphi}\left[\frac{\partial}{\partial r} + \frac{\imath}{2\pi r}\frac{\partial}{\partial \varphi}\right].$$

- The condition of analyticity:

$$\frac{\partial}{\partial \bar{w}} f(w) = 0 \quad \Leftrightarrow \quad = \frac{\partial f}{\partial r}(\varphi, r) + \frac{\imath}{2\pi r}\frac{\partial}{\partial \varphi}(\varphi, r) = 0.$$

$$\frac{\partial}{\partial r}\hat{f}(n, r) - \frac{n}{r}\hat{f}(n, r) = 0, \quad \hat{f}(n, r) = f_n r^n.$$

Motivation

Cauchy-Riemann equations (cont.)

- $$f(\varphi, r) = \sum_{n\in\mathbb{Z}} f_n\, r^n\, e^{2\pi\imath\, n\,\varphi} = \sum_{n\in\mathbb{Z}} f_n \left(r\, e^{2\pi\imath\,\varphi} \right)^n,$$

  $$f(w) = \sum_{n\in\mathbb{Z}} f_n\, w^n.$$

- $$f \in \mathrm{Hol}(\mathbb{D}) \quad \Leftrightarrow \quad \sup_{\substack{0<r<r_0<1 \\ \varphi\in\mathbb{T}}} \left| f(r\, e^{2\pi\imath\,\varphi}) \right| < \infty \quad \Leftrightarrow \quad \mathrm{supp}\, f_\bullet \subset \mathbb{N}_0.$$

The General Scheme

The General Scheme

The geometric setting

- G Abelian Lie group, Y manifold
- Test functions
$$\mathcal{D}(G \times Y) = \mathcal{S}(G) \,\hat{\otimes}\, C_c^\infty(Y)$$
- Distrubtions
$$\mathcal{D}(G \times Y)' = \mathcal{S}(G)' \,\hat{\otimes}\, C_c^\infty(Y)'$$

The General Scheme

Fourier transform

- Half-Fourier transform

$$\mathfrak{F} : \mathcal{D}(G \times Y) \to \mathcal{D}(\hat{G} \times Y),$$

$$\mathfrak{F} \otimes \mathbf{1} : \mathcal{S}(G) \otimes C_c^\infty(Y) \to \mathcal{S}(\hat{G}) \otimes C_c^\infty(Y).$$

- Formally,

$$\mathfrak{F} f(\xi, y) = \hat{f}(\xi, y) = \int_G f(x, y) e^{-2\pi \imath \langle \xi, x \rangle} dx$$

- By duality,

$$\mathfrak{F} : \mathcal{D}(G \times Y)' \to \mathcal{D}(\hat{G} \times Y)'.$$

The General Scheme

Intermediate constructions

- Multiplier algebra

$$\mathcal{O}_M(G \times Y) \times \mathcal{D}(G \times Y) \to \mathcal{D}(G \times Y),$$

$$\mathcal{O}_M(G \times Y) \times \mathcal{D}(G \times Y)' \to \mathcal{D}(G \times Y)'.$$

- G-invariant differential operators $\mathrm{D}_G(G \times Y)$

- G-moderate differential operators

$$\mathrm{D}_M(G \times Y) = \langle \mathcal{O}_M(G \times Y), \mathrm{D}_G(G \times Y) \rangle.$$

The General Scheme

Measurable functions on $G \times Y$

- Invariant measure on G:

$$d\mu(x, y) = dx d\nu(y), \quad d\nu \in \Lambda^{\dim Y}(Y).$$

- Invariant measure on $\hat{G}$:

$$d\hat{\mu}(\xi, y) = d\xi d\nu(y)$$

- Lebesgue spaces

$$L^p(G \times Y) \doteq L^p(G \times Y, \mu)$$

The General Scheme

Mixed-norm spaces

- Relatively arbitrary Banach spaces

$$\mathcal{Y}(Y) \hookrightarrow C_c^\infty(Y)', \quad \Xi(\hat{G}) \hookrightarrow \mathcal{S}(\hat{G})', \quad \Xi(\hat{G}) \subset L^1_{\mathrm{loc}}(\hat{G}).$$

- Mixed-norm space

$$\Xi(\hat{G}, \mathcal{Y}(Y)) = \left\{ \hat{u} \in L^1_{\mathrm{loc}}(\hat{G}, \mathcal{Y}(Y)) \,\middle|\, \|\hat{u}(\cdot)\|_{\mathcal{Y}} \in \Xi(\hat{G}) \right\},$$

$$\|\hat{u}\|_{\mathcal{Y},\Xi} = \|\|\hat{u}(\cdot)\|_{\mathcal{Y}}\|_{\Xi}$$

- Under reasonable assumptions $(\Xi(\hat{G}, \mathcal{Y}(Y)), \|\cdot\|_{\mathcal{Y},\Xi})$ normed vector space,

$$\Xi(\hat{G}, \mathcal{Y}(Y)) \hookrightarrow \mathcal{D}(\hat{G} \times Y)'$$

The General Scheme

Mixed-Fourier-norm spaces

- Mixed-Fourier-norm space

$$\mathcal{X}(G \times Y) = \left\{ u \in \mathcal{D}(G \times Y)' \,\middle|\, \hat{u} = \mathfrak{F} u \in \Xi(\hat{G}, \mathcal{Y}(Y)) \right\},$$

$$\|u\| = \|\hat{u}\|_{\mathcal{Y},\Xi}$$

- Isometric isomorphism of normed vector spaces

$$\mathfrak{F} : \mathcal{X}(G \times Y) \to \Xi(\hat{G}, \mathcal{Y}(Y))$$

- Is $\mathcal{X}(G \times Y)$ complete?

The General Scheme

Complex structure on $G \times Y$

- $\dim G = n$, $Y \subset \mathbb{R}^n$ open.
- $z = x + \imath y \in G + \imath Y$.
- Factorisation of holomorphic functions:

$$u \in \mathrm{Hol}(G \times Y) \quad \Leftrightarrow \quad \hat{u}(\xi, y) = e^{-2\pi \langle \xi, y \rangle} \hat{u}_0(\xi)$$

Remember $\hat{f}(n, r) = f_n r^n$, $r = e^{-2\pi y}$!

The General Scheme

Intermediate constructions

- Weight function

$$\rho(\xi) = \left\| e^{-2\pi\langle\cdot,\xi\rangle} \right\|_{\mathcal{Y}}$$

- Weighted space

$$\Xi(\hat{G},\rho) = \left\{ \hat{u}_0 \in L^1_{\mathrm{loc}}(\hat{G}) \,\middle|\, |\hat{u}_0|\rho \in \Xi(\hat{G}) \right\},$$

$$\|\hat{u}_0\|_{\Xi,\rho} = \||\hat{u}_0|\rho\|_{\Xi}$$

- Under reasonable assumptions, $(\Xi(\hat{G},\rho), \|\cdot\|_{\Xi,\rho})$ a Banach space.

The General Scheme

Holomorphic mixed-Fourier-norm spaces

- Holomorphic mixed-Fourier-norm space

$$\mathcal{A}_{\mathcal{X}}(G \times Y) = \mathcal{X}(G \times Y) \cap \mathrm{Hol}(G \times Y)$$

Proposition

Under mild assumptions,

$$\mathfrak{F} : \mathcal{A}_{\mathcal{X}}(G \times Y) \to e^{-2\pi\langle \cdot, \cdot \rangle} \cdot \Xi(\hat{G}, \rho), \quad u(z) \mapsto e^{-2\pi\langle \xi, y \rangle}\, \hat{u}_0(\xi)$$

is an isometric isomorphism of Banach spaces.

The General Scheme

Analytic factorisation

- Ω complex manifold, G acts analytically on Ω.
- Principal stratum: $\mathring{\Omega}_{\{0\}} \subset \Omega$ dense.
- G-equivariant bi-holomorphism $\Phi : G \times Y \to \mathring{\Omega}_{\{0\}}$.
- Assume that $Y \subset (\mathbb{R}_+)^n$ open, $n = \dim G$.

The General Scheme

Paley-Wiener-type properties

- Under mild assumptions,

$$\operatorname{supp} \hat{u}_0 \subset \hat{G} \cap [0, +\infty)^n, \quad \forall u \in \mathcal{A}_{\mathcal{X}}(G \times Y)$$

- Under mild assumptions, $\exists y_0 \in (\mathbb{R}_+)^n$ such that $\forall y_1 > y_0$,

$$\sup_{\substack{y > y_1 \\ x \in G}} \|u(x, y)\| < \infty, \quad \forall u \in \mathcal{A}_{\mathcal{X}}(G \times Y)$$

Examples

Examples

Unit disc: elliptic geometry

- $\Omega = \mathbb{D}, \quad G = \mathbb{T}, \quad \mathbb{T} \times \mathbb{D} \ni (x, w) \mapsto w\, e^{2\pi\imath x} \in \mathbb{D}$
- $\Omega_{\{0\}} = \mathring{\Omega}_{\{0\}} = \mathbb{D} \setminus \{0\}, \quad Y = \mathbb{R}_+$
- $\Phi : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{D} \setminus \{0\}, \quad \Phi(z) = \Phi(x + \imath y) = e^{2\pi z} = r\, e^{2\pi\imath x}$
- $$w \to \partial\mathbb{D} \quad \Leftrightarrow \quad z \to \mathbb{T} \times \{0\}$$
  $$w \to 0 \quad \Leftrightarrow \quad z \to +\imath\infty$$

Examples

Unit disc: elliptic geometry (cont.)

- $\hat{G} = \mathbb{Z}, \quad \Xi(\hat{G}) = l^q,\ q \in [1, +\infty), \quad \mathcal{Y}(Y) = X(\mathbb{R}_+)$
- $u = f \circ \Phi, \qquad u \in C_c^\infty(\mathbb{T} \times \mathbb{R}_+), \quad f \in C_c^\infty(\mathbb{D} \setminus \{0\}),$

$$\hat{u}(\xi, x) = \hat{f}(\xi, r) = \int_{\mathbb{T}} f(re^{2\pi\imath x}) e^{-2\pi\imath\xi x} dx$$

- $r^* X((0,1)) = X(\mathbb{R}_+), \quad h \in X((0,1)) \quad \Leftrightarrow \quad h \circ \exp \in X(\mathbb{R}_+)$

Examples

Unit disc: elliptic geometry (cont.)

- Mixed-Fourier-norm space $\mathcal{L}^{q;X}(\mathbb{D}) = \Phi_* \mathcal{X}(\mathbb{T} \times \mathbb{R}_+)$,

$$\mathcal{L}^{q;X}(\mathbb{D}) = \left\{ f \in C_c^\infty(\mathbb{D} \setminus \{0\})' \,\middle|\, \|f\| < \infty \right\},$$

$$\|f\| = \left(\sum_{\xi \in \mathbb{Z}} \|\hat{f}(\xi, \cdot)\|^q_{X((0,1))} \right)^{\frac{1}{q}}$$

- Bergman space $\mathcal{A}^{q;X}(\mathbb{D}) = \mathcal{L}^{q;X}(\mathbb{D}) \cap \mathrm{Hol}(\mathbb{D} \setminus \{0\})$,

$$\mathcal{A}^{q;X}(\mathbb{D}) = \left\{ f \in \mathrm{Hol}(\mathbb{D} \setminus \{0\}) \,\middle|\, \hat{f}(\xi, r) = f_\xi r^\xi, \quad \|f\| < \infty \right\},$$

$$\|f\| = \left(\sum_{\xi \in \mathbb{Z}} |f_\xi|^q \|r^\xi\|^q_{X((0,1))} \right)^{\frac{1}{q}}$$

Examples

Half-plane: hyperbolic geometry

- $\Omega = \Pi = \mathbb{C}_+, \quad G = \mathbb{R}, \quad \mathbb{R} \times \Pi \ni (x, w) \mapsto w\, e^x \in \Pi$
- $\Omega_{\{0\}} = \mathring{\Omega}_{\{0\}} = \Pi, \quad Y = (0, \pi)$
- $\Phi : \Gamma = \mathbb{R} \times (0, \pi) \to \Pi, \quad \Phi(z) = \Phi(x + \imath y) = e^z = r\, e^{\imath y}$
- $$w \to \partial\Pi \setminus \{0\} \quad \Leftrightarrow \quad z \to \partial\Gamma$$
  $$w \to \infty \cdot \mathbb{S}_+ \quad \Leftrightarrow \quad z \to +\infty$$
  $$w \to 0 \quad \Leftrightarrow \quad z \to -\infty$$

Examples

Half-plane: hyperbolic geometry (cont.)

- $\hat{G} = \mathbb{R}, \quad \Xi(\hat{G}) = L^q(\mathbb{R}), q \in [1, +\infty), \quad \mathcal{Y}(Y) = X((0, \pi))$

- $\mathrm{T}_\lambda : C(\Pi) \to C(\Gamma), \quad \lambda > -1$

$$\mathrm{T}_\lambda f(z) = e^{(\frac{\lambda}{2}+1)z} f(e^z)$$

- $u = \mathrm{T}_\lambda f, \quad u \in \mathcal{D}(\Gamma), \quad f \in \mathcal{D}(\Pi),$

$$\hat{f}(\xi, y) = e^{-y(1+\frac{\lambda}{2})} \hat{u}(\xi, x) = \int_0^\infty f(r, y) r^{-2\pi\imath\xi + \frac{\lambda}{2}} dr$$

Examples

Half-plane: hyperbolic geometry (cont.)

- Mixed-Fourier-norm space $\mathcal{L}^{q;X}(\Pi) = \mathrm{T}_\lambda^{-1}\,\mathcal{X}(\Gamma)$,

$$\mathcal{L}^{q;X}(\Pi) = \left\{ f \in \mathcal{D}(\Pi)' \,\middle|\, \quad \|f\| < \infty \right\},$$

$$\|f\| = \left(\int_{-\infty}^{\infty} \|\tilde{f}(\xi,\cdot)\|_{X((0,\pi))}^{q} \right)^{\frac{1}{q}}$$

- Bergman space $\mathcal{A}^{q;X}(\Pi) = \mathcal{L}^{q;X}(\Pi) \cap \mathrm{Hol}(\Pi)$,

$$\mathcal{A}^{q;X}(\Pi) = \left\{ f \in \mathrm{Hol}(\Pi) \,\middle|\, \quad \tilde{f}(\xi, y) = \tilde{f}_0(\xi) e^{-2\pi\xi y}, \quad \|f\| < \infty \right\},$$

$$\|f\| = \left(\int_{-\infty}^{\infty} |\tilde{f}_0(\xi)|^q \|e^{-2\pi\xi\cdot}\|_{X((0,\pi))}^{q} \right)^{\frac{1}{q}}$$

Thank you.

Z. A., A. Karapetyans “Mixed-Fourier-norm spaces and holomorphic functions”.

Arxiv: 2411.15379