Homogeneous Operators and Homogeneous Integral Operators

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Motivation

Motivation

Classical integral operators with homogeneous kernels on $\mathbb{R}_+$

$$\mathrm{K}\, f(x) = \int\limits_{\mathbb{R}_+} k(x,y) f(y) dy$$

$$\left(\forall (x,y) \in \mathbb{R}_+^2\right) (\forall \lambda \in \mathbb{R}_+) \quad k(\lambda x, \lambda y) = \frac{1}{\lambda} k(x,y).$$

More precisely,

$$\left(\text{a.e.}\, (x,y) \in \mathbb{R}_+^2\right) (\forall \lambda \in \mathbb{R}_+) \quad k(\lambda x, \lambda y) = \frac{1}{\lambda} k(x,y).$$

Motivation

Classical integral operators with homogeneous kernels on $\mathbb{R}^n$

$$\mathrm{K}\, f(x) = \int_{\mathbb{R}^n} k(x,y) f(y) dy$$

$$\left(\forall (x,y) \in \mathbb{R}^{2n}\right) (\forall \lambda \in \mathbb{R}_+) \quad k(\lambda x, \lambda y) = \frac{1}{\lambda^n} k(x,y),$$
$$\left(\forall (x,y) \in \mathbb{R}^{2n}\right) (\forall \omega \in \mathrm{SO}(n)) \quad k(\omega x, \omega y) = k(x,y).$$

Or rather

$$\left(\text{a.e.}\, (x,y) \in \mathbb{R}^{2n}\right) (\forall (\lambda, \omega) \in \mathbb{R}_+ \times \mathrm{SO}(n)) \quad k(\lambda \omega x, \lambda \omega y) = \frac{1}{\lambda^n} k(x,y).$$

Motivation

What good is the homogeneity of the kernel for?

- Dilations: constant Jacobian,

$$d(\lambda x) = \lambda dx.$$

- Homogeneous kernel: cancels the Jacobian,

$$k(\lambda x, \lambda y) d(\lambda y) = k(x, y) dy.$$

- **Strong** homogeneity of the kernel:

$$\left(\text{a.e. } (x, y) \in \mathbb{R}_+^2\right) (\forall \lambda \in \mathbb{R}_+) \quad k(\lambda x, \lambda y) = \frac{1}{\lambda} k(x, y).$$

Motivation

What good is the homogeneity of the kernel for?

- Ultimate purpose (on the level of the operator K):

$$(\forall\lambda \in \mathbb{R}_+) \quad \int_{\mathbb{R}_+} k(\lambda x, \lambda y) f(y) d(\lambda y) = \int_{\mathbb{R}_+} k(x, y) f(y) dy.$$

- **Weak** homogeneity of the kernel:

$$(\forall\lambda \in \mathbb{R}_+) \left(\text{a.e.}\,(x, y) \in \mathbb{R}_+^2\right) \quad k(\lambda x, \lambda y) = \frac{1}{\lambda} k(x, y).$$

The general theory

The general theory

Measure dilations

- M - measurable space (abstract, topological, manifold etc.)
- $\mathrm{Aut}(M)$ - automorphisms (measurable, homeomorphism, diffeomorphism etc.)

Definition

Let (M, μ) be a measure space. A transformation $\varphi \in \mathrm{Aut}(M)$ is a dilation if

$$(\exists \lambda_\varphi > 0) \quad \mu \circ \varphi = \lambda_\varphi \cdot \mu,$$

or

$$(\exists \lambda_\varphi > 0)\,(\forall A \in \Sigma_M) \quad \mu(\varphi(A)) = \lambda_\varphi \cdot \mu(A).$$

The general theory

The setting

- (M, μ) measure space
- G - group of dilations of (M, μ)
- $\varphi \mapsto g, \quad \varphi(x) \mapsto gx$ for $x \in M$
- G acts transitively on M
- $\lambda : G \to \mathbb{R}_+$ character, $\quad \mu(gA) = \lambda_g \mu(A)$ for $A \in \Sigma_M$
- $g \in \ker \lambda \subset G$ are measure-preserving, $\mu(gA) = \mu(A)$

Weak and strong homogeneity

Definition

A measurable kernel function $k \in L(M^2, \mu^{\otimes 2})$ is called:

- **Weakly** homogeneous if

$$(\forall g \in G) \left(\mu^{\otimes 2}\text{-a.e.}\,(x, y) \in M^2\right) \quad k(gx, gy) = \frac{1}{\lambda_g} k(x, y)$$

- **Strongly** homogeneous if

$$\left(\mu^{\otimes 2}\text{-a.e.}\,(x, y) \in M^2\right) (\forall g \in G) \quad k(gx, gy) = \frac{1}{\lambda_g} k(x, y)$$

The general theory

Homogeneous space

Transitivity $\Rightarrow M \simeq G/H, \quad H \subset G$.

- **Case A:** $H \subset \ker \lambda$ or $\lambda|_H = 1$.
- **Case B:** $H \not\subset \ker \lambda$ or $\lambda|_H \neq 1$.

Case A allows a reduction to the case $\lambda = 1$, i.e., G-invariant measure on G/H and convolutions.

The general theory

Case A

- $G \ni g \to gH = x \in G/H$ canonical quotient map
- $\lambda|_H = 1$ implies

$$\left(\exists\tilde{\lambda} : G/H \to \mathbb{R}_+\right) (\forall g \in G) \quad \lambda_g = \tilde{\lambda}(gH)$$

- Consider $(G/H, \tilde{\mu})$, where

$$d\tilde{\mu}(x) = \frac{1}{\tilde{\lambda}(x)} d\mu(x), \quad x \in G/H$$

- Then $\tilde{\mu}$ is G-invariant,

$$d\tilde{\mu}(gx) = d\tilde{\mu}(x)$$

The general theory

Case A (cont.)

For $p > 0$, $\mathrm{U}_p : L^p(G/H, \mu) \to L^p(G/H, \tilde{\mu})$ unitary,

$$\mathrm{U}_p f(x) = \tilde{\lambda}(x)^{\frac{1}{p}} f(x), \quad \forall x \in G/H, \quad \forall f \in L^p(G/H, \mu).$$

Quasiregular representation $\mathrm{L}_g : L(G/H, \mu) \to L(G/H, \mu)$,

$$\mathrm{L}_g f(x) = f(g^{-1}x), \quad \forall x \in G/H, \quad \forall g \in G, \quad \forall f \in L(G/H, \mu).$$

- $\mathrm{L}_g : L^p(G/H, \tilde{\mu}) \to L^p(G/H, \tilde{\mu})$ unitary
- $\mathrm{U}_p \mathrm{L}_g = \lambda_g^{\frac{1}{p}} \mathrm{L}_g \mathrm{U}_p$, $g \in G$

The general theory

The geometry of dilations

Define the measurable set $\mathcal{X}_H^\lambda \subset G/H$ as

$$\mathcal{X}_H^\lambda \doteq \{x \in G/H| \quad (\exists h \in H)\, hx = x \wedge \lambda_h \neq 1\}\,.$$

- Case A: $\mathcal{X}_H^\lambda = \emptyset$, regular kernels
- Case B: $\mathcal{X}_H^\lambda \neq \emptyset$, singular kernels

Further,

- $M_* \doteq G/H \setminus \mathcal{X}_H^\lambda$ - regular part of $G/H = M$
- $\mathrm{p}_H : G/H \to H\backslash G/H$ canonical quotient, $H\backslash M_* \doteq \mathrm{p}_H(M_*)$

The general theory

The geometry of dilations (cont.)

- $\mathcal{T} \to H\backslash M_*$ line bundle, $\wp : L(\mathcal{T}) \hookrightarrow L(M_*, \mu)$ embedding
- representative functions

$$\mathcal{F}_H^\lambda \doteq \left\{ F \in L(G/H, \mu) \middle| \quad (\forall x \in G/H)\,(\forall h \in H)\, F(hx) = \frac{1}{\lambda_h} F(x) \right\}.$$

Proposition

$$\mathcal{F}_H^\lambda = \left\{ F \in L(G/H, \mu) \middle| \quad F|_{\mathcal{X}_H^\lambda} = 0, \quad F|_{M_*} \in \wp(L(\mathcal{T})) \right\}.$$

The general theory

Homogeneous operators

Definition

Let $\mathcal{D} \subset L(G/H, \mu)$ be a vector subspace, $\mathrm{K} : \mathcal{D} \to L(G/H)$ a linear operator. We will call K homogeneous if

$$\mathrm{L}_g(\mathcal{D}) \subset \mathcal{D},$$

$$\mathrm{K}\,\mathrm{L}_g\, f = \mathrm{L}_g\, \mathrm{K}\, f, \quad \forall f \in \mathcal{D}, \quad \forall g \in G.$$

The general theory

Homogeneous operators: Case A

In Case A: for $p > 0$,

$$\widetilde{\mathcal{D}}_p \doteq \mathrm{U}_p(\mathcal{D}), \quad \mathrm{L}_g(\widetilde{\mathcal{D}}_p) \subset \widetilde{\mathcal{D}}_p, \quad \forall g \in G,$$

$$\widetilde{\mathrm{K}}_p \doteq \mathrm{U}_p\,\mathrm{K}\,\mathrm{U}_p^{-1} : \widetilde{\mathcal{D}}_p \to L(G/H, \tilde{\mu}),$$

$$\mathrm{L}_g\,\widetilde{\mathrm{K}}_p f = \widetilde{\mathrm{K}}_p\,\mathrm{L}_g\,f, \quad \forall f \in \widetilde{\mathcal{D}}_p, \quad \forall g \in G.$$

Thus, $\widetilde{\mathrm{K}}_p$ is a G-invariant operator on G/H.

The general theory

Homogeneous integral operators: week homogeneity

- Integral kernel $k \in L([G/H]^2, \mu^{\otimes 2})$
- Integral operator $\mathrm{K} : \mathcal{D} \to L(G/H, \mu), \quad \mathcal{D} \subset L(G/H, \mu)$,

$$\mathrm{K}\, f(x) = \int\limits_{G/H} k(x,y) f(y) d\mu(y), \quad \mu\text{-a.e. } x \in G/H, \quad \forall f \in \mathcal{D}.$$

- Homogeneous integral operator = integral operator that is homogeneous.

The general theory

Homogeneous integral operators: week homogeneity (cont.)

A vector subspace $\mathcal{D} \subset L(G/H, \mu)$ will be said to separate points if $\exists \{f_k\}_{k=1}^{\infty} \subset \mathcal{D}$ such that

$$(\forall F \in L(G/H, \mu)) \left[(\forall k \in \mathbb{N}) \int\limits_{G/H} F(x) f_k(x) d\mu(x) = 0 \right] \quad \Rightarrow \quad F = 0.$$

Theorem

Assume that $\mathcal{D} \subset L(G/H, \mu)$ separates points and $\mathrm{L}_g(\mathcal{D}) \subset \mathcal{D}$, $g \in G$. An integral operator $\mathrm{K} : \mathcal{D} \to L(G/H, \mu)$ is homogeneous if and only if its integral kernel k is **weakly** homogeneous.

The general theory

Homogeneous integral kernels: strong homogeneity

Theorem

A measurable kernel function $k \in L([G/H]^2, \mu^{\otimes 2})$ is **strongly** homogeneous if and only if

$$\left(\exists F \in \mathcal{F}_H^\lambda\right) \left(\forall (aH, bH) \in [G/H]^2\right) \quad k(aH, bH) = \frac{1}{\lambda_a} F(a^{-1} bH).$$

Thus, in Case B, every homogeneous kernel is expected to demonstrate exceptional behaviour around the singular set $\mathcal{X}_H^\lambda$.

Examples

Examples

1. Cylinder $\mathbb{R} \times \mathbb{T}$

- $M = G = \mathbb{R} \times \mathbb{T}, \quad g = (a, \varphi), \quad x = (z_x, \theta_x)$
- $gx = (a, \varphi)(z_x, \theta_x) = (z_x + a, \theta_x + \varphi \,\mathrm{mod}\, 2\pi)$
- $d\mu(z_x, \theta_x) = e^{2z_x} dz_x d\theta_x, \quad \lambda_{(a,\theta)} = e^{2a}$
- Strong homogeneity condition:

$$k(z_x + a, \theta_x + \varphi \mod 2\pi; z_y + a, \theta_y + \varphi \mod 2\pi) = e^{-2a} k(z_x, \theta_x; z_y, \theta_y)$$

- General form:

$$k(z_x, \theta_x; z_y, \theta_y) = e^{-z_x - z_y} F(z_x - z_y, \theta_x - \theta_y + 2\pi \mod 2\pi)$$

Examples

2. Plane $\mathbb{R}^2$

- $M = \mathbb{R}^2 \setminus \{0\}, \quad G = \mathbb{R} \times \mathbb{T}, \quad g = (a, \varphi), \quad x = (r_x, \theta_x)$
- $gx = (a, \varphi)(r_x, \theta_x) = (e^a r_x, \theta_x + \varphi \bmod 2\pi)$
- $d\mu(r_x, \theta_x) = r_x dr_x d\theta_x, \quad \lambda_{(a,\varphi)} = e^{2a}$
- Strong homogeneity condition:

$$k(e^a r_x, \theta_x + \varphi \text{ mod } 2\pi; e^a r_y, \theta_y + \varphi \text{ mod } 2\pi) = e^{-2a} k(r_x, \theta_x; r_y, \theta_y)$$

- General form:

$$k(r_x, \theta_x; r_y, \theta_y) = \frac{1}{r_x r_y} F\left(\frac{r_x}{r_y}, \theta_x - \theta_y + 2\pi \text{ mod } 2\pi\right)$$

Examples

2.1 Hadamard-Bergman convolution operators

- $\mathbb{D} \subset \mathbb{C} = \mathbb{R}^2, \quad d\mu(z) = \pi^{-1} dz_1 dz_2,$

$$\mathrm{K}\, f(z) = \int\limits_{\mathbb{C}} g(w) f(z\bar{w}) d\mu(w)$$

- Substitution $\xi = z\bar{w}$ yields

$$\mathrm{K}\, f(z) = \frac{1}{|z|^2} \int\limits_{|z|\cdot\mathbb{D}} g\left(\frac{\bar{\xi}}{\bar{z}}\right) f(\xi) d\mu(\xi)$$

- This corresponds to

$$\mathrm{K}\, f(z) = \int\limits_{\mathbb{C}} k(z, w) f(w) d\mu(w), \quad z \in \mathbb{D},$$

Examples

2.1 Hadamard-Bergman convolution operators (cont)

- where $z = |z|e^{\imath\theta_z} = (|z|, \theta_z), \quad w = |w|e^{\imath\theta_w} = (|w|, \theta_w)$,

$$k(z, w) = k(|z|, \theta_z; |w|, \theta_w) = \frac{1}{|z|^2} \begin{cases} g\left(\frac{|w|}{|z|} e^{\imath(\theta_z - \theta_w)}\right) & \text{if} \quad |w| < |z|, \\ 0 & \text{else} \end{cases}$$

- or

$$F(r, \theta) = \frac{1}{r} \begin{cases} g\left(\frac{1}{r} e^{\imath(\theta_z - \theta_w)}\right) & \text{if} \quad r > 1, \\ 0 & \text{else} \end{cases} .$$

Examples

3. Disk in $\mathbb{R}^2$ with radial measure

- $M = (0, R) \times \mathbb{T}, \quad G = \mathbb{R} \times \mathbb{T}, \quad g = (a, \varphi), \quad x = (r_x, \theta_x)$
- $d\mu(r_x, \theta_x) = \pi^{-1}\gamma(r_x^2) r_x dr_x d\theta_x, \quad \lambda_{(a,\varphi)} = e^{2a}$
- $gx = (a, \varphi)(r_x, \theta_x) = (r_*(r_x; a), \theta_x + \varphi \bmod 2\pi)$
- For all $C \in \mathbb{R}$ such that $\Gamma_C : (0, R^2) \to (0, +\infty)$,

$$\Gamma_C(t) = \int^t \gamma(s) ds + C$$

- General form:

$$k(r_x, \theta_x; r_y, \theta_y) = \frac{1}{\sqrt{\Gamma_C(r_x^2)\Gamma_C(r_y^2)}} F\left(\sqrt{\frac{\Gamma_C(r_x^2)}{\Gamma_C(r_y^2)}}, \theta_x - \theta_y + 2\pi \mod 2\pi\right)$$

Examples

3.1 Poincaré disk $\mathbb{D} \subset \mathbb{C}$

- $M = (0,1) \times \mathbb{T} = \mathbb{D} \setminus \{0\}$,

$$d\mu(r_x, \theta_x) = \frac{r_x dr_x d\theta_x}{\pi(1 - r_x^2)^2}.$$

- Here

$$\gamma(t) = \frac{1}{(1-t)^2}, \quad \Gamma_C(t) = \frac{1}{1-t} + C, \quad t \in (0,1), \quad C \geq -1.$$

- Setting $z = r_x e^{\imath\theta_x}$, $w = r_y e^{\imath\theta_y}$ and $C = -1$,

$$k(z,w) = \frac{\sqrt{(1-|z|^2)(1-|w|^2)}}{|zw|} G\left(\frac{|z|\sqrt{1-|w|^2}}{|w|\sqrt{1-|z|^2}}, \frac{z\bar{w}}{|zw|} \right).$$

Examples

3.2 $\mathbb{D} \subset \mathbb{C}$ with weighted Bergman measure

- $M = (0,1) \times \mathbb{T} = \mathbb{D} \setminus \{0\}, \quad \alpha \in (-1, +\infty)$,

$$d\mu(r_x, \theta_x) = \frac{(\alpha+1)(1-r_x^2)^\alpha r_x dr_x d\theta_x}{\pi}.$$

- Here

$$\gamma(t) = (\alpha+1)(1-t)^\alpha, \quad \Gamma_C(t) = -(1-t)^{\alpha+1} + C, \quad t \in (0,1), \quad C \geq 1.$$

- Setting $z = r_x e^{\imath\theta_x}$, $w = r_y e^{\imath\theta_y}$,

$$k(z,w) = \frac{1}{\sqrt{(C-(1-|z|^2)^{\alpha+1})(C-(1-|w|^2)^{\alpha+1})}} \times$$

$$G\left(\sqrt{\frac{C-(1-|z|^2)^{\alpha+1}}{C-(1-|w|^2)^{\alpha+1}}}, \frac{z\bar{w}}{|zw|}\right).$$

Examples

4 $\mathrm{GL}(n)$-homogeneous integral kernels on $\mathbb{R}^n$, $n > 1$

- $M = G/H = \mathbb{R}^n \setminus \{0\}, \quad G = \mathrm{GL}(n)$
- $H = \mathrm{Aff}(n-1) \simeq \mathrm{GL}(n-1) \rtimes \mathbb{R}^{n-1}$

$$d\mu(x) = dx, \quad \lambda_g = |\det g|, \quad \forall g \in G.$$

- $\lambda|_H \neq 1$, Case B.
- $n > 2, \quad \mathcal{X}_H^\lambda = G/H,\, M_* = \emptyset, \quad k = 0$
- $n = 2,\, \mathcal{X}_H^\lambda = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \neq 0,\, x_2 = 0\}, \quad H \backslash M_* = \{e\}$

Examples

4 $\mathrm{GL}(n)$-homogeneous integral kernels on $\mathbb{R}^n$, $n > 1$ (cont.)

Unique (up to a factor) homogeneous kernel:

$$k(x,y) = \begin{cases} \frac{1}{|[x,y]|} & \text{for} \quad [x,y] \neq 0, \\ 0 & \text{else} \end{cases}, \quad [x,y] = x_1 y_2 - x_2 y_1.$$

$$\mathrm{K}\, f(x) = \int_{\mathbb{R}^2} \frac{f(y)}{|[x,y]|} dy$$

Integral converges conditionally only for a very narrow class of f.

Examples

4 $\mathrm{GL}^+(n)$-homogeneous integral kernels on $\mathbb{R}^n$, $\quad n > 1$

- $G = \mathrm{GL}^+(n), \quad H = \mathrm{Aff}^+(n-1) \simeq \mathrm{GL}^+(n-1) \rtimes \mathbb{R}^{n-1}$
- $n > 2, \quad \mathcal{X}_H^\lambda = G/H, M_* = \emptyset, \quad k = 0$
- $n = 2, \quad H \backslash M_* = \{a, b\}$

$$k(x,y) = \begin{cases} \frac{C_+}{[x,y]} & \text{for} \quad [x,y] > 0, \\ \frac{C_-}{[x,y]} & \text{for} \quad [x,y] < 0, \\ 0 \quad \text{else} & \end{cases}$$

Examples

4* $\mathrm{GL}^+(n)$-homogeneous integral kernels on $\mathbb{R}^n$, $n > 1$ (cont.)

Unique (up to a factor) antisymmetric homogeneous kernel:

$$k(x, y) = \begin{cases} \frac{1}{[x,y]} & \text{for} \quad [x, y] \neq 0, \\ 0 & \text{else} \end{cases} , \quad [x, y] = x_1 y_2 - x_2 y_1 .$$

$$\mathrm{K}\, f(x_1, x_2) = \int_{\mathbb{R}^2} \frac{f(y_1, y_2)}{x_1 y_2 - x_2 y_1} dy_1 dy_2$$

Formally, setting $x_2 = y_2 = 1$ (no integration in y_2),

$$\mathrm{K}\, f(x_1, 1) = \int_{\mathbb{R}} \frac{f(y_1, 1)}{x_1 - y_1} dy_1$$

Conclusion

Main results

- Dilations in measure space, the geometry of dilations
- Case A: reduction to G-invariance and convolution theory
- Case B: no recourse to invariance, only singular kernels
- Homogeneous integral operator $\Leftrightarrow$ weakly homogeneous kernel
- Strongly homogeneous kernel $\Leftrightarrow$ general formula
- Examples: cylinder, plane, Hadamard-Bergman, radial - Case A
- Example: $\mathrm{GL}(2)$, unique singular operator - Case B

Conclusion

Open questions

- Structure and regularization in Case B, general theory
- Properties of the unique operator in $\mathrm{GL}(2)$ case
- Appropriate choices $G \subsetneq \mathrm{GL}(n)$, $n > 2$
- Operator theory, function spaces and properties of general homogeneous integral operators

Thank you.

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