

Universal functions and sets

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PART I: Universal Functions

Definitions

Definitions

Universal sequence

- Ω a Hausdorff sequential convergence space
- \mathcal{X} a Banach space, with a continuous and dense embedding in $\mathcal{X} \hookrightarrow \Omega$
- $\{\varphi_n\}_{n=1}^{\infty}$ a minimal system in \mathcal{X}

Definition

A sequence $\{\xi_n\}_{n=1}^{\infty} \subset \mathcal{X}$ is called **universal** in Ω if subsequences of $\{\xi_n\}_{n=1}^{\infty}$ converge to every point of Ω ,

$$(\forall f \in \Omega) (\exists \{N_k\}_{k=1}^{\infty} \subset \mathbb{N}) \quad \xi_{N_k} \xrightarrow[k \rightarrow \infty]{\Omega} f.$$

Definitions

Reductions

- $\mathcal{X} \hookrightarrow \Omega \hookrightarrow \Theta$ continuous dense embeddings
- \mathcal{X} a Banach space
- Ω, Θ Hausdorff first-countable topological spaces

Lemma

Suppose that the sequence $\{\xi_n\}_{n=1}^{\infty} \subset \mathcal{X}$ is universal in Ω . Then it is universal in Θ .

Definitions

Minimal systems and series

- \mathcal{X} a Banach space, \mathcal{X}^* its dual space
- $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{X}$ a minimal system
- $\{c_n\}_{n=1}^{\infty} \subset \mathcal{X}^*$ the dual system, $c_n(\varphi_m) = \delta_{n,m}$

Polynomial or series

$$\sum_{n=1}^N \alpha_n \varphi_n, \quad \alpha_n \in \mathbb{C}, \quad N \in \mathbb{N} \cup \{\infty\}$$

Fourier series of $f \in \mathcal{X}$

$$\sum_{n=1}^{\infty} c_n(f) \varphi_n$$

Definitions

Definition

A series $\sum_{n=1}^{\infty} \alpha_n \varphi_n$ is said to be:

1. **universal** in Ω in the **usual** sense if partial sums universal in Ω ,

$$(\forall f \in \Omega) (\exists \{N_k\}_{k=1}^{\infty} \subset \mathbb{N}) \quad \sum_{n=1}^{N_k} \alpha_n \varphi_n \xrightarrow[k \rightarrow \infty]{} f$$

2. **universal** in Ω in the sense of \mathbb{G} -values (**signs**) if

$$(\forall f \in \Omega) (\exists \{\epsilon_n\}_{n=1}^{\infty}) \quad (\forall n \in \mathbb{N}) \epsilon_n \in \mathbb{G} \quad \wedge \quad \sum_{n=1}^{\infty} \epsilon_n \alpha_n \varphi_n = f$$

3. **universal** in Ω in the sense of **rearrangements** if

$$(\forall f \in \Omega) (\exists \sigma \in \text{Aut}(\mathbb{N})) \quad \sum_{n=1}^{\infty} \alpha_{\sigma(n)} \varphi_{\sigma(n)} = f$$

Definitions

Definition

We will call an element $U \in \mathcal{X}$:

1. **universal** for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ in the **usual** sense if the Fourier series $\sum_{n=1}^{\infty} c_n(U)\varphi_n$ is universal in Ω in the usual sense
2. **universal** for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ in the sense of **\mathbb{G} -values (signs)** if the Fourier series $\sum_{n=1}^{\infty} c_n(U)\varphi_n$ is universal in Ω in the sense of \mathbb{G} -values (signs)
3. **universal** for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ in the sense of **rearrangements** if the Fourier series $\sum_{n=1}^{\infty} c_n(U)\varphi_n$ is universal in Ω in the sense of rearrangements

Definitions

Definition

4. **conditionally universal** for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ if

$$(\exists\{\delta_n\}_{n=1}^{\infty}) \quad (\forall n \in \mathbb{N}) \delta_n \in \mathbb{G}$$

and the series $\sum_{n=1}^{\infty} \delta_n c_n(\mathbf{U}) \varphi_n$ is universal in Ω in the usual sense

5. **almost universal** for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ if

$$(\exists\{\delta_n\}_{n=1}^{\infty}) \quad (\forall n \in \mathbb{N}) \delta_n \in \mathbb{G} \quad \wedge$$

$$\limsup_{n \rightarrow \infty} \frac{\#\{m \in \mathbb{N} \mid m \leq n \wedge \delta_m = 1\}}{n} = 1$$

and the series $\sum_{n=1}^{\infty} \delta_n c_n(\mathbf{U}) \varphi_n$ is universal in Ω in the usual sense

Universality in metric spaces

Universality in metric spaces

The Universal Approximation Property

Definition

Let $\Omega = (\Omega, d)$ be a metric space. We will say that the minimal system $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{X}$ possesses the **universal approximation property** in Ω if for a dense subset $\mathcal{D} \subset \mathcal{X}$ we have

$$(\forall f \in \mathcal{D}) (\forall \epsilon, \delta > 0) (\forall n_0 \in \mathbb{N}) \\ (\exists N \in \mathbb{N}) \left(\exists \{\alpha_n\}_{n=n_0}^N \subset \mathbb{C} \right) \left(\exists \{\delta_n\}_{n=n_0}^N \subset \mathbb{G} \right)$$

such that:

1. $\|H\|_{\mathcal{X}} < \epsilon$, $H \doteq \sum_{n=n_0}^N \alpha_n \varphi_n$
2. $d(f, Q) < \delta$, $Q \doteq \sum_{n=n_0}^N \delta_n \alpha_n \varphi_n$

Universality in metric spaces

Abstract theorem

- (Ω, d) a separable Abelian metric group
- $(\mathcal{X}, \|\cdot\|)$ a Banach space
- $\mathcal{X} \hookrightarrow \Omega$ a continuous dense additive embedding
- $(\{\varphi_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty})$ a bi-orthonormal system in $(\mathcal{X}, \mathcal{X}^*)$

Theorem

If the system $\{\varphi_n\}_{n=1}^{\infty}$ possesses the universal approximation property in Ω then $\exists U \in \mathcal{X}$ such that U is almost universal for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$.

Asymptotic universality in L^1

Asymptotic universality in L^1

Definition

Let $\{\varphi_n\}_{n=1}^\infty \subset L^1(\mathcal{M})$ be a minimal system. $U \in L^1(\mathcal{M})$ is:

1. **asymptotically universal** for $L^1(\mathcal{M})$ w.r.t. $\{\varphi_n\}_{n=1}^\infty$ in the **usual** sense if there exists a sequence of subsets $\{F_m\}_{m=1}^\infty \subset 2^{\mathcal{M}}$, with

$$F_1 \subset F_2 \subset \dots \subset \mathcal{M}, \quad \lim_{m \rightarrow \infty} |F_m^c| = 0, \quad (1)$$

such that

$$\left(\forall f \in L^1(\mathcal{M}) \right) \left(\exists \{N_q\}_{q=1}^\infty \subset \mathbb{N} \right) \left(\forall m \in \mathbb{N} \right)$$

$$\lim_{q \rightarrow \infty} \int_{F_m} \left| \sum_{n=1}^{N_q} c_n(U) \varphi_n(x) - f(x) \right| dx = 0$$

Asymptotic universality in L^1

Definition

2. **asymptotically conditionally universal** for $L^1(\mathcal{M})$ w.r.t. $\{\varphi_n\}_{n=1}^\infty$ if there exist a sequence of \mathbb{G} -values (signs) $\{\delta_n\}_{n=1}^\infty$, $(\forall n \in \mathbb{N}) \delta_n \in \mathbb{G}$, and a sequence of subsets $\{F_m\}_{m=1}^\infty \subset 2^{\mathcal{M}}$, with

$$F_1 \subset F_2 \subset \dots \subset \mathcal{M}, \quad \lim_{m \rightarrow \infty} |F_m^c| = 0, \quad (2)$$

such that

$$\left(\forall f \in L^1(\mathcal{M}) \right) \left(\exists \{N_q\}_{q=1}^\infty \subset \mathbb{N} \right) \left(\forall m \in \mathbb{N} \right)$$

$$\lim_{q \rightarrow \infty} \int_{F_m} \left| \sum_{n=1}^{N_q} \delta_n c_n(U) \varphi_n(x) - f(x) \right| dx = 0$$

Asymptotic universality in L^1

Definition

We will say that the minimal system $\{\varphi_n\}_{n=1}^\infty \subset L^1(\mathcal{M})$ possesses the **asymptotic approximation property** in $L^1(\mathcal{M})$ if for a dense subset $\mathcal{D} \subset L^1(\mathcal{M})$ and a positive number $C > 0$ we have

$$(\forall f \in \mathcal{D}) (\forall \epsilon, \delta, \sigma > 0) (\forall n_0 \in \mathbb{N}) \\ (\exists N \in \mathbb{N} + n_0) \left(\exists \{\alpha_n\}_{n=n_0}^N \subset \mathbb{C} \right) \left(\exists \{\delta_n\}_{n=n_0}^N \subset \mathbb{G} \right) (\exists E \subset \mathcal{M}) \quad (3)$$

such that:

1. $|E^c| < \sigma$
2. $\|H\|_1 < \epsilon$, $H \doteq \sum_{n=n_0}^N \alpha_n \varphi_n$
3. $\int_E |f(x) - Q(x)| dx < \delta$, $Q \doteq \sum_{n=n_0}^N \delta_n \alpha_n \varphi_n$
4. $\int_{E^c} |Q(x)| dx \leq C \cdot \|f\|_1$

Existence of asymptotically conditionally universal functions

Theorem

If the minimal system $\{\varphi_n\}_{n=1}^{\infty}$ possesses the asymptotic approximation property in $L^1(\mathcal{M})$ then there exists an integrable function $U \in L^1(\mathcal{M})$ which is asymptotically conditionally universal for $L^1(\mathcal{M})$.

Asymptotic universality in L^1

Abundance of asymptotically conditionally universal functions

Theorem

If the minimal system $\{\varphi_n\}_{n=1}^{\infty}$ possesses the asymptotic approximation property in $L^1(\mathcal{M})$ then there exist an integrable function $U \in L^1(\mathcal{M})$, which is asymptotically conditionally universal for $L^1(\mathcal{M})$ w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$, and a sequence of subsets $\{E_m\}_{m=1}^{\infty} \subset 2^{\mathcal{M}}$, with

$$E_1 \subset E_2 \subset \dots \subset \mathcal{M}, \quad \lim_{m \rightarrow \infty} |E_m^c| = 0,$$

such that

$$\left(\forall g \in L^1(\mathcal{M}) \right) \left(\forall m \in \mathbb{N} \right) \left(\exists V_m \in L^1(\mathcal{M}) \right)$$

$$\left(\forall x \in E_m \right) \quad V_m(x) = g(x)$$

$$\wedge \left(\exists \{\varepsilon_n\}_{n=1}^{\infty} \subset \mathbb{G} \right) \left(\forall n \in \mathbb{N} \right) \quad c_n(V_m) = \varepsilon_n c_n(U).$$

Examples

Examples

The trigonometric system in $L^p([-\pi, \pi])$, $p \in [0, 1)$

Lemma (Galoyan, Grigoryan'21)

Let $\Omega = L^p([-\pi, \pi])$, $p \in (0, 1)$, $\mathcal{X} = L^1([-\pi, \pi])$ and consider the trigonometric system

$$\varphi_n(x) = e^{inx}, \quad \forall x \in [-\pi, \pi], \quad \forall n \in \mathbb{N}.$$

Then the system $\{\varphi_n\}_{n=1}^{\infty}$ possesses the universal approximation property in Ω .

Corollary

There exists a function $U \in L^1([-\pi, \pi])$ which is almost universal for $L^p([-\pi, \pi])$, $p \in [0, 1)$, as well as $M([-\pi, \pi])$, w.r.t. the trigonometric system.

Examples

The trigonometric system in $L^1([-\pi, \pi])$

Lemma (Galoyan, Grigoryan'18)

The trigonometric system $\{\varphi_n\}_{n=1}^{\infty}$ possesses the asymptotic approximation property in $L^1([-\pi, \pi])$.

Corollary

There exists a function $U \in L^1([-\pi, \pi])$ which is asymptotically conditionally universal for $L^1([-\pi, \pi])$ w.r.t. the trigonometric system.

Corollary

For every $\epsilon > 0$ there exists a set E with $|E^c| < \epsilon$, such that for every function $f \in L^1([-\pi, \pi])$ one can find a function $U \in L^1([-\pi, \pi])$ with $U|_E = f|_E$, which is asymptotically conditionally universal for $L^1([-\pi, \pi])$ w.r.t. the trigonometric system.

PART II: Universal Sets

Measure space

Approximations

- Norm approximations: given f find g s.t.

$$\|f - g\| < \epsilon$$

- Spatial approximations: given f find g s.t.

$$(f - g)|_E = 0, \quad |E^c| < \epsilon$$

Theorem (Luzin'1912)

For every $f : [a, b] \rightarrow \mathbb{C}$ measurable, a.e. finite, and $\epsilon > 0$ there exist $g \in C[a, b]$ and $E_{f,\epsilon} \subset [a, b]$ s. t.

$$(f - g)|_{E_{f,\epsilon}} = 0, \quad |E_{f,\epsilon}^c| < \epsilon.$$

Fourier series in L^1

- $\mathcal{M} = (\mathcal{M}, \Sigma, \mu)$ finite separable measure space
- $\{\varphi_n\}_{n=1}^{\infty}$ ONB in $L^2(\mathcal{M})$, $\varphi_n \in L^\infty(\mathcal{M})$ for all $n \in \mathbb{N}$
- For $f \in L^1(\mathcal{M})$,

$$c_n(f) = (f, \varphi_n)_2 = \int_{\mathcal{M}} f(x) \varphi_n^*(x) d\mu(x), \quad Y_n(x; f) = c_n(f) \varphi_n(x)$$

- **Problem:** For which $f \in L^1(\mathcal{M})$ does the Fourier series converge?

$$\sum_{n=1}^{\infty} Y_n(\cdot; f) \rightarrow f \quad \text{in } L^1(\mathcal{M})$$

Spatial approximations in L^1

Theorem (Grigoryan'91)

Let

$$\varphi_n(x) = e^{inx} \in L^\infty[-\pi, \pi].$$

For $\forall \epsilon > 0$ there is a subset

$$E_\epsilon \subset [-\pi, \pi] \quad \text{with} \quad |E_\epsilon| > 2\pi - \epsilon$$

with the following property; for every $f \in L^1[-\pi, \pi]$ there is a $g \in L^1[-\pi, \pi]$ such that

$$(f - g)|_{E_\epsilon} = 0 \quad \text{and} \quad \sum_n Y_n(\cdot, g) \rightarrow g \quad \text{in} \quad L^1[-\pi, \pi].$$

Cylindric structure

Obstacles against generalization to $\{\varphi_n\}_{n=1}^{\infty}$ and \mathcal{M} :

- Uses special properties of e^{inx}
- Uses the structure of $[a, b] \subset \mathbb{R}$

Possibilities:

- Use measure theoretic isomorphism $\mathcal{M} \simeq [a, b]$
- Find natural cylindric structure $\mathcal{M} = [a, b] \times \mathcal{N}$

The main result

Theorem:

Let \mathcal{M} be a separable finite diffuse measure space, and let $\{\varphi_n\}_{n=1}^{\infty}$ be an ONB in $L^2(\mathcal{M})$ with $\varphi_n \in L^\infty(\mathcal{M})$. For every $\epsilon, \delta > 0$ there is a subset $E_\epsilon \subset \mathcal{M}$ with $|E_\epsilon| > |\mathcal{M}| - \delta$ and with the following property; for each function $f \in L^1(\mathcal{M})$ with $\|f\|_1 > 0$ there exists a $g \in L^1(\mathcal{M})$ s.t.

- $\|f - g\|_1 < \epsilon,$
- $(f - g)|_{E_\epsilon} = 0,$

-

$$\sum_{n=1}^{\infty} Y_n(\cdot; g) \rightarrow g \in L^1(\mathcal{M}),$$

-

$$\sup_m \left\| \sum_{n=1}^m Y_n(\cdot; g) \right\|_1 < 2 \min\{\|f\|_1, \|g\|_1\}.$$

Homogeneous Space

Example: sphere

- $\mathcal{M} = \mathbb{S}^2$, $x = (\theta, \varphi)$,

$$d\mu(x) = \frac{1}{4\pi^2} \sin \theta d\theta d\varphi.$$

- Label $n = (\rho, m)$, $\rho \in \mathbb{N}_0$, $m = -\rho, \dots, \rho$,

$$\varphi_n(x) = Y_{\rho}^m(\theta, \varphi) = \sqrt{\frac{(2\rho + 1)(\rho - m)!}{(\rho + m)!}} P_{\rho}^m(\cos \theta) e^{im\varphi}.$$

- Cylindric structure $\mathbb{S}^2 \simeq [0, 1] \times [0, 1]$,

$$(\theta, \varphi) \mapsto \left(\frac{\varphi}{2\pi}, \frac{\theta}{\pi} \right).$$

Invariant measure on G/H

- G - infinite compact second countable Hausdorff group
- $H \subset G$ - closed subgroup, G/H - infinite homogeneous space
- $x \in G/H$ means $x = xH$, where $x \in G$
- dx - Haar measure on G , dh - Haar measure on H
- dx - unique normalized G -invariant Radon measure on G/H such that

$$\int_G f(x) dx = \int_{G/H} \left(\int_H f(xh) dh \right) dx, \quad \forall f \in C(G).$$

Fourier transform on G/H

- \hat{G} - unitary dual of G
- $d_\rho^H = \text{mult}(\mathbf{1}, \rho|_H)$ for $\forall \rho \in \hat{G}$

$$\widehat{G/H} = \left\{ \rho \in \hat{G} \mid d_\rho^H > 0 \right\}$$

$$\mathbb{P}_H = \int_H \rho(h) dh, \quad \rho(x) = \rho(x)\mathbb{P}_H, \quad \forall \rho \in \widehat{G/H}, \quad \forall x = xH \in G/H$$

- **Fourier** transform of $f \in L^1(G/H)$ is

$$\hat{f}(\rho) = \int_{G/H} f(x) \rho(x)^* d\mu(x)$$

Fourier series on G/H

- Block Fourier series of $f \in L^1(G/H)$,

$$\sum_{\rho \in \widehat{G/H}} d_\rho \operatorname{tr} \left[\widehat{f}(\rho) \rho(x) \right], \quad d_\rho = \dim \mathcal{H}_0 \rho \in \mathbb{N}$$

- Choose $\{\mathbf{e}_i^\rho\}_{i=1}^{d_\rho}$ ONB in \mathcal{H}_ρ , and $\{\mathbf{e}_\alpha^\rho\}_{\alpha=1}^{d_\rho^H}$ ONB in $\mathbb{P}_H \mathcal{H}_\rho$

- Label $n = (\rho, i, \alpha)$, $\varphi_n(x) = \sqrt{d_\rho} (\rho(x) \mathbf{e}_\alpha^\rho, \mathbf{e}_i^\rho)$,

$$\mathbf{c}_{\rho, i, \alpha}(f) = (f, \varphi_{\rho, i, \alpha})_2, \quad Y_{\rho, i, \alpha}(x; \rho) = \mathbf{c}_{\rho, i, \alpha}(f) \varphi_{\rho, i, \alpha}(x)$$

- Then

$$\operatorname{tr} \left[\widehat{f}(\rho) \rho(x) \right] = \sum_{i=1}^{d_\rho} \sum_{\alpha=1}^{d_\rho^H} Y_{\rho, i, \alpha}(x; \rho)$$

Main theorem for G/H

Theorem:

Let $\mathcal{M} = G/H$ be an infinite homogeneous space of a compact second countable Hausdorff group G with closed isotropy subgroup $H \subset G$.

For every $\epsilon, \delta > 0$ there is a subset $E_\epsilon \subset G/H$ with $|E_\epsilon| > 1 - \delta$ and with the following property; for each function $f \in L^1(G/H)$ with $\|f\|_1 > 0$ there exists a $g \in L^1(G/H)$ s.t.

- $\|f - g\|_1 < \epsilon,$
- $(f - g)|_{E_\epsilon} = 0,$

-

$$\sum_{\rho \in \widehat{G/H}} d_\rho \operatorname{tr} [\hat{g}(\rho)\rho(x)] \rightarrow g \in L^1(G/H),$$

-

$$\sup_{\rho_0 \in \widehat{G/H}} \left\| \sum_{\rho \leq \rho_0} d_\rho \operatorname{tr} [\hat{g}(\rho)\rho(x)] \right\|_1 < 2 \min\{\|f\|_1, \|g\|_1\}.$$

Cylindric structure in G/H

- $K \subset G$ infinite closed subgroup,

$$G/H^{(K)} = \{x \in G/H \mid \exists 1 \neq k \in K \text{ s.t. } kx = x\} \subset G/H$$

- $q : G/H \rightarrow K \backslash G/H, \quad q(x) = Kx$
- $\nu = \mu \circ q^{-1}$ probability measure on $K \backslash G/H$

Proposition

If $|G/H^{(K)}| = 0$ then there exists a Borel almost isomorphism $\varphi : K \times K \backslash G/H \rightarrow G/H$ such that

$$\varphi(\tilde{k} \cdot k, Kx) = \tilde{k}\varphi(k, Kx), \quad q(\varphi(k, Kx)) = Kx,$$

$$\forall k, \tilde{k} \in K, \quad \forall Kx \in K \backslash G/H.$$

Construction

The construction of E_ϵ and g

- Choose an arbitrary ordering $\{R_k\}_{k=1}^\infty$ of all Fourier polynomials with rational coefficients.
- For every $k \in \mathbb{N}$, choose a partition $\{\Delta_l(k)\}_{l=1}^{\nu_0(k)}$ of the cylindrical measure space $\mathcal{M} = [0, 1] \times \mathcal{N}$ of the form

$$\Delta_l(k) = [a_l(k), b_l(k)] \times \tilde{\Delta}_l(k),$$

such that the measures $\|\Delta_l(k)\|$ are small enough, as well as a subordinate real step function

$$\Lambda(k) = \sum_{l=1}^{\nu_0(k)} \gamma_l(k) \chi_{\Delta_l(k)},$$

such that $\|\Lambda(k) - R_k\|_1$ is sufficiently small.

The construction of E_ϵ and g (2)

- For every $k \in \mathbb{N}$, choose a number $\delta_*(k) \in (0, \frac{1}{2})$ so that $\{\delta_*(k)\}_{k=1}^\infty$ decays sufficiently rapidly. Define the periodic step function

$$l(t) = 1 - \frac{1}{\delta_*(k)} \chi_{[0, \delta_*(k))}(t \bmod 1),$$

and the measurable function $\hat{g}_l^k \in L^\infty(\mathcal{M})$ by

$$g_l^k(x) = \gamma_l(k) l(s_0(k)t) \chi_{\Delta_l(k)}(x), \quad \forall x = (t, y) \in \mathcal{M} = [0, 1] \times \mathcal{N},$$

where the positive number $s_0(k)$ is sufficiently large.

The construction of E_ϵ and g (3)

- Define the measurable subsets $\hat{E}_l(k) \subset \Delta_l(k)$ by

$$\hat{E}_l(k) = \left\{ x \in \Delta_l(k) \mid g_l^k(x) = \gamma_l(k) \right\}.$$

Define inductively the natural numbers $\hat{N}_l(k)$, $l = 0, \dots, \nu_0(k)$ and Fourier polynomials \hat{Q}_l^k , $l = 1, \dots, \nu_0(k)$, by setting $\hat{N}_0(1) = 1$, $\hat{N}_0(k) = \hat{N}_{\nu_0(k)}(k-1)$ for $k > 1$, and $\hat{Q}_l^k = \sum_{n=\hat{N}_{l-1}(k)}^{\hat{N}_l(k)-1} Y_n(\hat{g}_l^k)$, so that the quantities

$$\left\| \sum_{n=1}^{\hat{N}_l(k)-1} Y_n(\hat{g}_l^k) - \hat{g}_l^k \right\|_2$$

are sufficiently small.

The construction of E_ϵ and g (4)

- For every $k \in \mathbb{N}$, define the natural numbers $N_k = \hat{N}_{\nu_0(k)} - 1$, measurable subsets $\tilde{E}_k = \bigcup_{l=1}^{\nu_0(k)} \hat{E}_l(k)$, and measurable functions $\tilde{g}_k \in L^\infty(\mathcal{M})$ by

$$\tilde{g}_k = R_k - \Lambda(k) + \sum_{l=1}^{\nu_0(k)} \hat{g}_l^k,$$

as well as Fourier polynomials

$$\tilde{Q}_k = \sum_{l=1}^{\nu_0(k)} \hat{Q}_l^k = \sum_{n=N_{k-1}}^{N_k-1} \tilde{Y}_n.$$

- Set

$$E = \bigcap_{k=1}^{\infty} \tilde{E}_k.$$

The construction of E_ϵ and g (5)

- Choose a subsequence $\{R_{k_s}\}_{s=0}^\infty$ such that $\|R_{k_s}\|_1$ decay sufficiently rapidly, and $\sum_{s=0}^\infty R_{k_s} = f$ in $L^1(\mathcal{M})$.
- Define inductively the sequence of natural numbers $\{\nu_s\}_{s=1}^\infty$, $\nu_s > \nu_{s-1}$ for $s > 1$, and measurable functions $g_s \in L^\infty(\mathcal{M})$ by choosing ν_1 so that $N_{\nu_1-1} > \max \sigma(R_{k_0})$ and

$$\left\| R_{\nu_s} - R_{k_s} + \sum_{j=1}^{s-1} [\tilde{Q}_{\nu_j} - g_j] \right\|_1$$

is sufficiently small, and setting $g_s = R_{k_s} + \tilde{g}_{\nu_s} - R_{\nu_s}$.

- Finally, set

$$g = R_{k_0} + \sum_{s=1}^{\infty} g_s.$$

Thank you.

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